PLANE AND SOLID GEOMETRY

SCHULTZE-SEVENOAK-SCHUYLER

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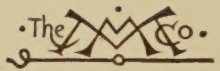
PLANE AND SOLID GEOMETRY



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PLANE AND SOLID GEOMETRY

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PREFACE TO THE FIRST EDITION

It is generally conceded that the final aim of mathematical teaching should be not only the acquisition of practical knowledge but that training of the student's mind which gives a distinct gain in mental power. In recognition of this principle nearly all college entrance examinations in geometry require some original work, and most textbooks devote considerable space to exercises. Comparatively little, however, has been done to introduce the student *systematically* to original geometrical work. No teacher of physics or chemistry would ask a student to discover a law without so guiding his work as to enable him to reach the desired result; many textbooks and teachers expect the pupil to invent geometrical proofs and to solve problems, entirely new to him, without offering any assistance further than a knowledge of the well-established theorems of all textbooks. Some writers give a description of the analysis of propositions, which is entirely logical and of great advantage to a person of some mathematical knowledge, but which is usually too abstract to be of any practical value to the beginner. In this book the attempt is made to introduce the student systematically to the solution of geometrical exercises. In the beginning the exercises given in a certain group are of similar kind and related to the preceding proposition; later some general principles are developed which are of fundamental importance for original work, as, for example, the method of proving the equality of lines by means of equal triangles; the method of proving the proportionality of lines by means of similar triangles, etc.; and

finally the analyses of theorems and problems are introduced, but in a more concrete form than usual.

The propositions are arranged with the view to obtaining a perfect logical and pedagogical order. An unusually large number of exercises is given, selected with care for the purpose of securing increased mental power.

The general plan and the preparation of the greater part of the book are the work of Dr. Schultze, while that of Dr. Sevenoak has been chiefly editorial.

PREFACE TO THE FIRST REVISED EDITION

THE main purpose of the revision of this book has been to emphasize still further and to elaborate in greater detail the principal aim of the original edition, viz., *to introduce the student systematically to original geometric work.* To make the teaching of geometry both disciplinary and informational; to give to the student mental training instead of teaching him mere facts; to develop his power instead of making him memorize, — these are the fundamental aims of this book.

The means employed for this purpose are similar to those used in the first edition. Still greater emphasis, however, has been placed upon the general methods which may be used for the solution of original exercises. The grading and the selection of exercises have been carefully revised. All originals that appeared unfit or too difficult have been eliminated or replaced by simpler and better ones. Topics of fundamental importance, *e.g.* the methods of demonstrating the equality of lines, are represented in greater detail and illustrated by a greater number of exercises than in the first edition.

In addition to these fundamental tendencies, a number of minor improvements have been introduced, among which may be mentioned:

Improved presentation of the regular propositions. Many proofs have been simplified, a more pedagogic sequence of the propositions of Book I has been adopted, Books VI and VII have been considerably simplified, and a number of difficult theorems of minor importance have been omitted or placed in the appendix.

Simplification of the so-called "incommensurable case." As this is a claim that is made by most textbooks, it may be received with some degree of skepticism, but a repeated trial of this new method will reveal its simplicity. For the more conservative teacher, however, who dislikes fundamental changes, the time-honored method is given in the appendix.

The introduction of many applied problems. These problems have been selected and arranged so as to increase the interest of the student, without sacrificing in the least the disciplinary value of the subject. Many such problems are given in the appendix.

The arrangement of the propositions and the terminology are in accord with the best modern usage. Thus, statements and reasons have been separated and placed in parallel vertical columns; the term "congruent" and the corresponding symbol are introduced and applied consistently, etc.

Many of the diagrams have been improved. The construction lines are drawn completely for most problems, graphical modes are employed for pointing out important facts, and many diagrams have been otherwise improved.

Thanks are due to Dr. J. Kahn and Mr. W. S. Schlauch for assistance in reading the proof and for helpful suggestions.

A. S.

August 1, 1913.

PREFACE TO THE PRESENT EDITION

In the twelve years since the first revision of Schultze and Sevenoak's Geometry was made, the progress of teaching and the codifying of mathematical requirements have necessitated some changes in form and content. The present revision follows closely the first revision in organization, but has been completely rewritten with additional material. The noteworthy points are:

1. The text has been made to conform to the "Report of the National Committee on Mathematical Requirements" and to the requirements of the New York State Regents and of the College Entrance Examination Board.

2. Most of the excellent exercises of the previous edition have been kept intact and other exercises, from recent examination questions, have been added.

3. The propositions, especially in Books I and II, have been rather fully developed.

4. Additions have been made in (1) propositions, (2) simple trigonometric functions, (3) tables, (4) short review section on arithmetic and algebra, (5) a sketch of the history of geometry.

5. At the end of the various books and in the appendix, the more difficult propositions have been retained so as to meet the most exacting requirements of any college or technical school.

The authors and publishers are conscious of a great debt of gratitude to many teachers who have aided in this revision by their criticisms and suggestions. While a full list of such

collaborators is impossible, special thanks are due the following: Mr. Stephen Emery, Erasmus Hall High School, Dr. Loring B. Mullen, Girls High School, Miss Josephine D. Wilkin, Jamaica High School, Mr. Philip R. Dean, Evander Childs High School, and Mrs. Jean F. Brown, Hunter College High School, of New York City; Mr. Edmund D. Searls, High School, of New Bedford, Massachusetts; Miss A. Laura Batt, High School, of Somerville, Massachusetts; Mr. Allen H. Knapp, Central High School, and Mr. Harry B. Marsh, Technical High School, of Springfield, Massachusetts; and Miss Anna H. Andrews, Public High School, of Hartford, Connecticut.

E. S.

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SUGGESTIONS TO STUDENTS

The student should set to work guided by the following suggestions:

1. Have materials — pencil, paper, ruler, compasses — ready for use.

2. Have a definite time and place to do your work.

3. Do your work *yourself*. Do it *regularly* and *conscientiously*.

4. Frequently review important matters already gone over to fix them firmly in your mind.

5. In a notebook classify your results, stating conditions that make angles equal, lines parallel, triangles equal, etc. Classify formulas.

6. Do not merely skim through the text. Give it thought. Read carefully, and thoroughly digest what you do.

SYMBOLS AND ABBREVIATIONS

$+$ . . plus, or added to.
$-$ . . minus, or diminished by.
$=$ . . equals, or is equivalent to.
$\cong$ . . congruent.
$\neq$ . . is not equal to.
$>$ . . is greater than.
$<$ . . is less than.
$\therefore$ . . therefore, or hence.
$\perp$ . . perpendicular, or is perpendicular to.
$\perp s$ . . perpendiculars.
$\parallel$ . . parallel, or is parallel to.
$\parallel s$ . . parallels.
$\sim$ . . is similar to, or similar.
$\angle$ . . angle.
$\angle s$ . . angles.
$\triangle$ . . triangle.
$\triangle s$ . . triangles.
$\square$ . . parallelogram.
$\square s$ . . parallelograms.
$\odot$ . . circle.
$\odot s$ . . circles.
$\frown$ . . arc; as $\overset{\frown}{AB}$, arc AB.
adj. . adjacent.

alt. . . . . alternate.
ax. . . . . axiom.
circum. . . circumference.
comp. . . complement.
con. . . . construction.
cor. . . . corollary.
corr. . . . corresponding.
def. . . . definition.
ex. . . . . exercise.
ext. . . . exterior.
hom. . . . homologous.
hy. . . . . hypotenuse.
hyp. . . . hypothesis.
iden. . . . identity.
int. . . . interior.
isos. . . . isosceles.
rt. . . . . right.
st. . . . . straight.
sub. . . . substitution.
sup. . . . supplementary, or supplement.

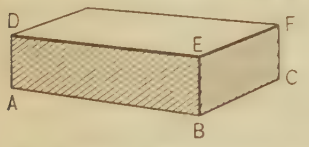
PLANE AND SOLID GEOMETRY

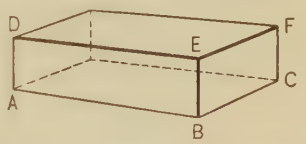
PLANE GEOMETRY

INTRODUCTION

DEFINITIONS

1. A physical body, such as a block of wood or iron, occupies a definite portion of space. The portion of space occupied by a physical body is called a *geometric solid* or a *solid*.





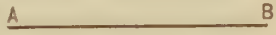
2. DEF. **Solids** are limited portions of space. They have three dimensions: *length*, *breadth*, and *thickness*.

3. DEF. **Surfaces** are the boundaries of solids; as *ABED* or *BEFC*. They have two dimensions: *length* and *breadth*.

The boundary between a window pane and the air is a surface. Obviously such a boundary has no thickness.

4. DEF. **Lines** are the boundaries of surfaces; as *AB*, *AD*. (Figure of § 1.) Lines have but one dimension: *length*.

Thus, the annexed black line *AB* is not a geometric line, for it has breadth.

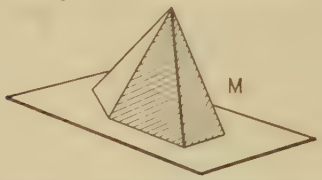


A true geometric line, however, is represented by the boundary between the black and the white.

5. DEF. **Points** are the boundaries, or the extremities, of lines. They are without dimensions, having position only.

Surfaces may be conceived as existing independently of the solids whose boundaries they form. In like manner, lines and points may exist independently in space.

6. DEF. A **geometric figure** is a point, line, surface, or solid, or any combination of any or all of these; as M or N.



A *rectilinear figure* is a figure composed of straight lines only.

7. DEF. **Geometry** is the science that treats of the properties of geometric figures.

8. The simplest line is a **straight line.** It is *represented* approximately by a string stretched taut between two points; as AB. The word *line* is frequently used to denote a straight line.

The notion of a straight line is such a simple and fundamental one that it is practically impossible to give a good definition of it.

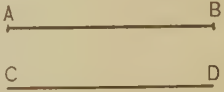
9. DEF. A **curved line** is a line no portion of which is straight; as CD in § 8.

10. DEF. A **broken line** is a line composed of different successive straight lines; as EF in § 8.

No *two successive* parts of a broken line lie in the same straight line.

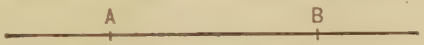
11. The expression, straight line, is used to denote both an unlimited straight line and a part of such line.

A *line of definite length*, also called a **segment**, or line-segment, is represented by a line whose ends are marked; as AB. The length of this line is also called the **distance** between A and B.



A line whose ends are not marked represents a line of indefinite length; as CD.

12. The **direction** of the line AB means the direction from A toward B; of BA, the direction from B toward A.



13. To **produce** the line AB means to prolong it through B; to **produce** BA means to prolong it through A.

14. DEF. A **plane surface**, or a **plane**, is a surface such that a straight line joining any two of its points lies entirely in the surface.

15. DEF. A **plane figure** is a geometric figure, all of whose points lie in the same plane; as *EDFG*.

16. DEF. **Plane Geometry** treats of plane figures only.

17. DEF. **Solid Geometry** treats of figures which are not plane. **Spherical Geometry** is a surface geometry.

18. When one figure can be placed upon another so that every point of each figure lies upon some point of the other, the figures are said to **coincide**.

19. DEF. **Congruent figures** are those that can be made to coincide.

For reasons that will appear later congruent lines are frequently called *equal lines*. Similarly, congruent angles are usually called *equal angles*. (See note, p. 205.)

20. Proof by superposition is the method of proving the congruence of two figures by making them coincide.

21. To **bisect** a line means to divide it into two equal parts.

Thus, AC is bisected if $AD = DC$. We assume that every line-segment (AC) has one and only one bisector. This bisector is a point; as D.

Ex. 1. What is the path of a moving point?

Ex. 2. What geometric figure is, in general, generated by a moving line? By a moving surface?

Ex. 3. Can a straight line move so that its path is not a surface?

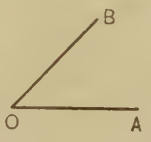
Ex. 4. How does a stone cutter use the straight edge to determine whether a surface is plane?

Ex. 5. What kind of surface is represented by each wall of a room?

Ex. 6. What kind of surface is represented by the outer surface of a gas pipe?

ANGLES

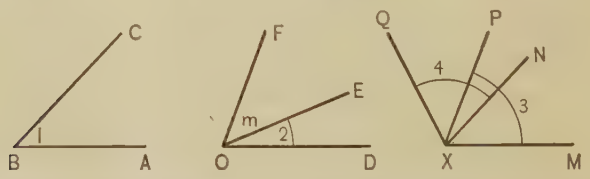
22. If a straight line OA revolves about one of its points O until the line reaches the position OB, then the amount of this rotation is called *the angle AOB*. Obviously the amount of rotation, and hence the angle, does not depend upon the length of the line which rotates.



The lines OA and OB are called the **sides** and the point O the **vertex** of the angle AOB.

We may define an *angle* to be the figure formed by two rays, or half lines, proceeding from a common point.

23. Notation. If three letters are used to denote an angle, the vertex letter should be read between the others; as angle ABC, angle EOF. A single letter at the vertex denotes the



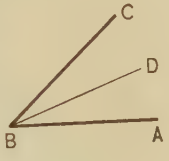
largest angle at this vertex (if there be several at this point). Thus, angle DOF may be read "angle O," angle ABC may be read "angle B."

Frequently an angle is also designated by a number, or italic letter, placed within it; as angle 1, angle 2, angle m.

Angle FOD is the sum of angles 2 and m. Angle 2 is the difference between angles FOD and m.

Often a curve is drawn to point out more clearly which angle is meant; as angle 2, and angle 3. An arc placed close to a number shows which angle is designated. Thus, angle MXP may be read "angle 3," and angle NXQ may be read "angle 4."

24. To **bisect** an angle means to divide it into two equal parts. We assume every angle has one and only one bisector.

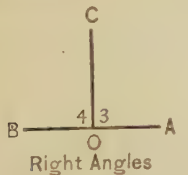


Thus, BD bisects angle ABC, if angle ABD = angle DBC. BD is called the **bisector** of angle B.

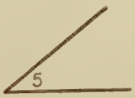
25. DEF. A **straight angle** is an angle whose sides lie in the same straight line but extend in opposite directions; as ABC.

26. DEF. A **right angle** is an angle equal to one half a straight angle.

Thus, if OC bisects the straight angle AOB, angle 3 and angle 4 are right angles.



Right Angles



Acute Angle

Obtuse Angle

27. DEF. An **acute angle** is an angle less than a right angle; as angle 5.

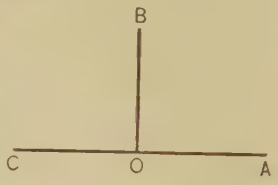
28. DEF. An **obtuse angle** is an angle greater than a right angle, but less than a straight angle; as angle MNO.

29. DEF. Acute and obtuse angles are called **oblique angles.**

30. DEF. Two lines are **perpendicular** to each other if they meet at right angles; as *AC* and *BO*.

The point of meeting (*O*) is the **foot** of the perpendicular.

The **distance** of a point from a line is the length of the perpendicular from the point to this line; as *BO*.



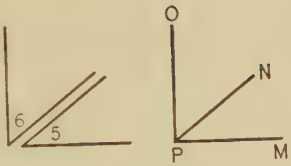
31. An angle is **measured** by finding how many times it contains a certain unit. The usual unit is the **degree**, or one ninetieth ($\frac{1}{90}$) of a right angle. A degree is divided into sixty equal parts called **minutes**, and a minute into sixty equal parts called **seconds**. Degrees, minutes, and seconds are expressed by symbols, as 6° 50′ 12″. Read *six degrees, fifty minutes, and twelve seconds.* Other units are the right angle and the straight angle.

32. DEF. **Adjacent angles** are two angles that have a *common vertex*, and a *common side between them;* as angles *AOB* and *BOC*.

33. DEF. Two angles are **vertical** (or **opposite**) **angles** if the sides of each are prolongations of the sides of the other back through their common vertex; as angles 7 and 8, or angles 9 and 10.

34. DEF. We can **add** two angles by placing them so that they are adjacent. The angle formed by the two sides not common is called their **sum.**

35. DEF. Two angles are **complementary** if their sum equals a right angle.



Each is then called the *complement* of the other. Angles 5 and 6, or MPN and NPO, are complementary.

36. Two angles are **supplementary** if their sum equals a straight angle (or two right angles).

Each angle is then called the supplement of the other. Angles 1 and 2, or angles 3 and 4, are supplementary.

Ex. 1. How many degrees are in a right angle? In a straight angle? In one half a right angle?

Ex. 2. What is the angle made by the two hands of a clock at three o'clock? At six o'clock? At two o'clock? At five o'clock?

Ex. 3. What is the angle made by the hands of a clock at 1 P.M.? At 2:30 P.M.? At 5:30 P.M.?

Ex. 4. Over an angle of how many degrees does a spoke of a wheel sweep when the wheel makes $\frac{1}{4}$ of a revolution? $\frac{1}{6}$ of a revolution? 2 revolutions?

Ex. 5. How large is each angle at the center if a pie is divided into 5 equal parts? 6 equal parts?

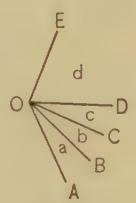
Ex. 6. What angle is formed by lines drawn north and northeast? Towards S. and S.E.? Towards N.W. and S.W.?

Ex. 7. Over what angle does the large hand of a watch sweep in 10 min.? 15 min.? 30 min.? 45 min.? 1 hr.?

Ex. 8. In diagram for Ex. 9 read by three letters: $\angle a$, $\angle b$, $\angle c$, $\angle d$, $\angle(a+b)$, $\angle(b+c+d)$.

Ex. 9. In diagrams similar to the one given here find the numerical values of the required angles:

(*a*) If $\angle a = 30°$, and $\angle b = 40°$, find $\angle AOC$.
(*b*) If $\angle b = 35°$, and $\angle c = 10°$, find $\angle BOD$.
(*c*) If $\angle b = 40°$, $\angle c = 10°$, and $\angle d = 50°$, find $\angle BOE$.
(*d*) If $\angle AOC = 60°$, and $\angle b = 40°$, find $\angle a$.
(*e*) If $\angle AOD = 90°$, $\angle a = 35°$, and $\angle c = 10°$, find $\angle b$.
(*f*) If $\angle AOE = 110°$, $\angle a = 20°$, and $\angle d = 30°$, find $\angle BOD$.
(*g*) If $\angle AOC = 60°$, and $\angle a = \angle b$, find $\angle a$.
(*h*) If $\angle AOD = 75°$, and $\angle a = \angle b = \angle c$, find $\angle c$.

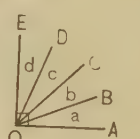


Ex. 9

Ex. 10. In the preceding diagram, which angles are adjacent to $\angle BOC$? To $\angle COD$? To $\angle BOD$?

Ex. 11. In diagrams similar to the one shown, if $\angle O = 90°$:

(*a*) Which angle is the complement of $\angle a$?
(*b*) Which angle is the complement of $\angle AOC$?
(*c*) Which angle is the complement of $\angle BOE$?
(*d*) If $\angle d = 20°$, find $\angle AOD$.
(*e*) If $\angle b = 20°$, and $\angle COE = 55°$, find $\angle a$.
(*f*) If $\angle AOC = 55°$, and $\angle d = 15°$, find $\angle c$.
(*g*) If $\angle a = \angle b = \angle c = \angle d$, find $\angle a$.



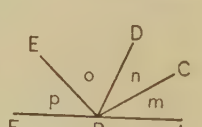
Ex. 11

Ex. 12. How many degrees are in the complement of 30°? Of 35°? Of $\frac{2}{3}$ of a right angle? Of $n°$? Of $\frac{1}{n}$ of a right angle? Of $(10 + x)°$?

Ex. 13. How many degrees are there in an angle that is twice its complement?

Ex. 14. In diagrams similar to the annexed one, if FBA is a straight line,

(*a*) Which angle is the supplement of $\angle p$?
(*b*) Which angle is the supplement of $\angle DBF$?
(*c*) Which angle is the supplement of $\angle ABE$?
(*d*) If $\angle p = 40°$, find $\angle ABE$.
(*e*) If $\angle m = 30°$, and $\angle p = 35°$, find $\angle CBE$.
(*f*) If $\angle DBF = 100°$, and $\angle m = \angle n$, find $\angle m$.
(*g*) If $\angle p = 30°$, and $\angle m = \angle n = \angle o$, find $\angle o$.
* (*h*) If $\angle FBC = 140°$, and $\angle ABD = 80°$, find $\angle n$.
* (*i*) If $\angle ABD = 80°$, $\angle n = 35°$, and $\angle CBE = 85°$, find $\angle p$.



Ex. 14

Ex. 15. How many degrees are in the supplement of 20°? Of 140°? Of $\frac{3}{4}$ of a straight angle? Of n degrees? Of $(50 - 3x)°$?

*Exercises denoted by (*) are difficult.

Ex. **16**. How many degrees are in an angle which is three times its supplement?

Ex. **17**. What kind of angle is less than its supplement? Equal to its supplement? Greater than its supplement?

Ex. **18**. Write in algebraic symbols:

(*a*) The complement of $n°$. (*b*) Three times the complement of $x°$. (*c*) The supplement of $(2x)°$. (*d*) Six times the supplement of $n°$.

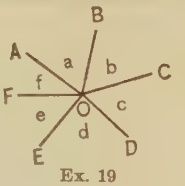
Ex. **19**. In diagrams similar to the annexed one find the numerical values of the required angles:

(*a*) If $\angle a = 80°$, $\angle b = 50°$, $\angle c = 60°$, $\angle d = 90°$, and $\angle e = 50°$, find $\angle f$.

(*b*) If $\angle a = \angle b = \angle c = \angle d = \angle e = \angle f$, find $\angle f$.

* (*c*) If $\angle AOC = 130°$, $\angle b = 50°$, $\angle BOD = 110°$, and $\angle DOF = 140°$, find $\angle f$.

(*d*) If $\angle d = 90°$, and $\angle c = \angle b = \angle a = \angle f = \angle e$, find $\angle a$.



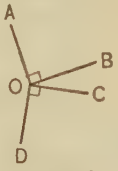
Ex. 19

Ex. **20**. If two lines, AB and CD, intersect in O, making $\angle AOC = 60°$, find the other angles.

Exs. 20, 21

Ex. **21**. If $\angle AOC = m$ degrees, how many degrees are in $\angle DOB$? In $\angle BOC$?

Ex. **22**. If $\angle AOB = \angle COD = 90°$, find $\angle AOD$ (*a*) if $\angle BOC = 60°$, (*b*) if $\angle BOC = m°$.



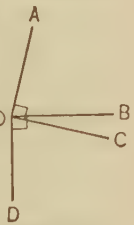
Exs. 22, 23

Ex. **23**. What relation exists between the angles BOC and AOD?

Ex. **24**. If AO is perpendicular to CO, and BO is perpendicular to DO, find $\angle AOD$, (*a*) if $\angle COB = 40°$, (*b*) if $\angle COB = m°$.

Ex. **25**. What relation exists between $\angle AOD$ and $\angle BOC$?

* Ex. **26**. If $\angle AOC = \angle BOD = 90°$, and $\angle AOD = 3\ \angle BOC$, find $\angle BOC$.



Exs. 24–26

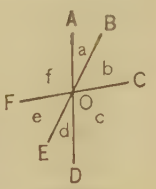
Ex. 27. Three lines meet in O forming the six angles: a, b, c, d, e, and f.

(a) If $\angle a = 20°$, and $\angle b = 60°$, find $\angle c$.

* (b) If $\angle a = 15°$, and $\angle c = 95°$, find $\angle e$.

* (c) If $\angle f = 100°$, and $\angle d = 20°$, find $\angle b$.

* (d) If $\angle AOC = 85°$, and $\angle BOD = 155°$, find $\angle e$.



Ex. 27

Ex. 28. Find the angle formed by the bisectors of the supplementary adjacent angles AOB and BOC, (a) if $\angle AOB = 40°$, (b) if $\angle AOB = 60°$, (c) if $\angle AOB = m°$.

Ex. 29. *What is the angle formed by the bisectors of any two supplementary adjacent angles?*

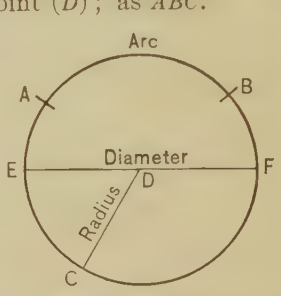
Ex. 30. Find the angle formed by the bisectors of the complementary adjacent angles AOB and BOC: (a) if $\angle AOB = 20°$, (b) if $\angle AOB = 30°$, (c) if $\angle AOB = m°$.

Ex. 31. What is the angle formed by the bisectors of any two complementary adjacent angles?

37. DEF. A **circle** is a plane closed curve, having all of its points equally distant from a fixed point (D); as ABC.

The **center** (D) is the fixed point. A **radius** is any straight line from the center to a point in the circle; as DC. An **arc** is any portion of the circle; as AB. The length of the circle is called the **circumference**.

The term *circumference* is frequently used to denote the curved line, and the term *circle* to denote the area. But modern usage is against this terminology.



38. Instruments used in geometry. Only two instruments are permitted to be used in plane geometry, viz., the *compasses* and the *straight edge*.

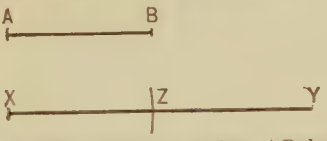
The straight edge is employed for the drawing of straight lines, the compasses for the drawing of circles or arcs and the transfer of lines of definite length (segments) from one position to another.[1]

[1] A line in a plane is said to be closed if it separates a finite portion of the plane from the remaining portion.

EXERCISES IN GEOMETRIC DRAWING

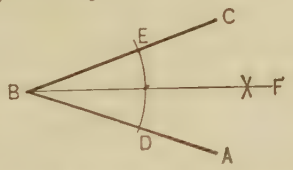
39. The following exercises are designed to familiarize the student with the use of the instruments and to impress upon him the meaning of the fundamental notions of geometry. They form, however, no logical part of plane geometry, and may be omitted without affecting the course. They are based upon three constructions, for which no proofs are given here. (Compare §§ 83 and 86.)

I. *To lay off on XY a line equal to AB.*



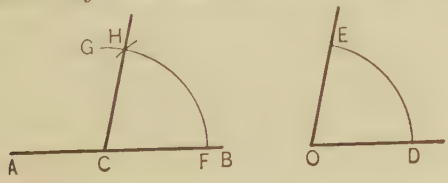
From X as a center with a radius equal to AB draw an arc intersecting XY in Z; XZ is the required line. (§ 38.)

II. *To bisect a given angle ABC.*



From B as a center with any radius draw an arc meeting AB in D, and AC in E. From D and E as centers with a radius sufficiently large draw two arcs intersecting in F. Then the line BF is the required bisector.

III. *From a point C in a given straight line AB, to draw a line to make an angle with AB equal to angle O.*



From O as a center, with any radius, describe an arc cutting the sides of $\angle O$ in D and E. From C as a center with the same radius describe arc FG, intersecting AB in F. From F as a center with a radius equal to DE,[1] draw an arc intersecting FG in H. Draw CH. Then $\angle BCH$ is the required angle.

Ex. **1.** Draw two points A and B (indicated by small dots or crosses) and construct a line through A and B.

Ex. **2.** Draw two points A and B, and construct a circle from A as a center with a radius equal to AB.

Ex. **3.** Draw three points A, B, and C, and construct a straight line through every pair of points.

Ex. **4.** Draw two lines of definite length, AB and CD, AB being the longer line, and construct lines equal to:

(*a*) $AB + CD$
(*b*) $AB - CD$
(*c*) $2(AB)$
(*d*) $AB + 2(CD)$
(*e*) $3(AB) - 2(CD)$

Ex. **5.** Draw an acute angle and bisect it.

Ex. **6.** Draw an obtuse angle and bisect it.

Ex. **7.** *Construct a right angle and bisect it.*

Ex. **8.** Draw a straight angle and bisect it.

Ex. **9.** Construct the sum of two angles.

Ex. **10.** At a given point C in AB draw a perpendicular to AB.

Ex. **11.** Divide a given angle into 4 equal parts.

Ex. **12.** Divide a given angle into 8 equal parts.

Ex. **13.** Construct an angle of: 90°; of 45°.

Ex. **14.** Construct an angle of: 22° 30′; of 135°.

Ex. **15.** Construct an angle of: 270°; of 67° 30′.

Ex. **16.** *Construct the supplement of a given angle A.*

Ex. **17.** Construct one half the supplement of a given angle A.

Ex. **18.** *Construct the complement of a given acute angle.*

Ex. **19.** Construct one half the complement of a given acute angle.

[1] Two letters, *e.g.* DE. used as above, denote a straight line.

Ex. 20. Construct the supplement of the complement of a given acute angle.

Ex. 21. Construct the complement of the supplement of a given obtuse angle.

Ex. 22. Draw two angles A and B, A being the greater one, and construct an angle equal to:

(a) $A + B$

(b) $2A$

(c) $180° - A$

(d) $90° + A$

(e) $A - B$

(f) $\frac{A}{2}$

(g) $\frac{A}{2} + B$

(h) $\frac{A + B}{2}$

(i) $\frac{A}{2} + \frac{B}{2}$

(j) $\frac{A}{2} - \frac{B}{2}$

(k) $90 + \frac{B}{2}$

(l) The complement of $\frac{B}{4}$

GENERAL TERMS

40. DEF. A **theorem** is a statement the truth of which is to be demonstrated (or proved).

Thus the statement: "*If two sides of a triangle are equal, the angles opposite are equal,*" is a theorem. The conditional part of the statement is called the *hypothesis*, the assertion that is to be proved is called the *conclusion*. Thus in the above example, the hypothesis is: "*If two sides of a triangle are equal,*" and the conclusion is: "*the angles opposite are equal.*" The hypothesis is, however, not always explicitly stated, but is sometimes merely implied.

41. DEF. A **problem**, in general, is a question to be solved.

The problems of geometry are solved either by constructions or by computations. Thus the problem: "To construct an angle of 45°" (Ex. 13, p. 12) is a *problem of construction.* "To compute the angle formed by the bisectors of a pair of complementary adjacent angles" (Ex. 30, p. 10) is a *problem of computation.*

42. DEF. A **proposition** is a general term for either a theorem or a problem.

43. DEF. An **axiom** is a statement whose truth is assumed.

Thus, the statement: "If equals be added to equals, the sums are equal," is an axiom.

44. DEF. A **postulate** is a purely geometric axiom.

Formerly elementary textbooks restricted the term *postulate* to "geometric constructions whose possibility is admitted without further demonstration," but modern usage favors the more general definition given above.

45. DEF. A **corollary** is a theorem easily derived from another theorem.

AXIOMS AND POSTULATES

1. *Things equal to the same thing, or to equal things, are equal to each other.*

2. *If equals are added to equals, the sums are equal.*

3. *If equals are subtracted from equals, the remainders are equal.*

4. *If equals are added to unequals, the sums are unequal in the same sense.*

5. *If equals are subtracted from unequals, the remainders are unequal in the same sense.*

6. *If unequals are subtracted from equals, the remainders are unequal in the opposite sense.*

7. *If equals are multiplied by equals, the products are equal.* (Important special case: *doubles of equals are equal.*)

8. *If equals are divided by equals, the quotients are equal.* (Important special case: *halves of equals are equal.*)

9. *If the first of three quantities is greater than the second, and the second is greater than the third, then the first is greater than the third.*

10. *The whole is equal to the sum of all its parts.*

11. *The whole is greater than any of its parts.*

12. *A quantity may be substituted for an equal one in an equation or in an inequality.* (Briefly called "*substitution.*")

13. *One straight line and only one can be drawn through two points (or two points determine a line).*

14. *A straight line is the shortest line between two points.*

15. *All straight angles are equal.*

16. *Two intersecting straight lines cannot both be parallel to a third straight line.*

17. *A straight line may be produced to any required length.*

18. *A circle may be drawn having any point as center and any line as radius.*

19. *A geometric figure may be moved from one position to another without change of form or size.*

The first twelve axioms are general axioms, the rest are geometric axioms or postulates.

A straight line is fully determined, given a point on this line and given the angle it makes with a given line.

From the thirteenth axiom these truths obviously follow:

(*a*) Two straight lines can intersect in only one point.

(*b*) Two lines of unlimited length, coinciding in part, coincide throughout.

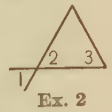
Ex. 1. Indicate the hypothesis and the conclusion of each of the following statements:

(*a*) If iron is heated, it expands.

(*b*) If two angles of a triangle are equal, the opposite sides are equal.

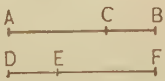
(*c*) Two triangles are congruent if the sides of the one are respectively equal to the sides of the other.

(*d*) Vertical angles are equal.



Ex. 2

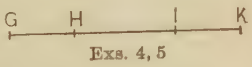
Ex. 2. If $\angle 1 = \angle 2$, and $\angle 2 = \angle 3$, for what reason [1] does $\angle 1 = \angle 3$?



Ex. 3

Ex. 3. If $AB = DF$, and $CB = DE$, for what reason does $AC = EF$?

Ex. 4. If $GH = IK$, for what reason does $GI = HK$?



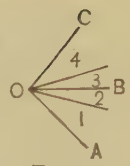
Exs. 4, 5

Ex. 5. If $GI = HK$, for what reason does $GH = IK$?

[1] The reasons requested in this and the following exercises are axioms.

Ex. **6**. If OB bisects $\angle O$, and $\angle 1 = \angle 4$, why does $\angle 2 = \angle 3$?

Ex. **7**. If $\angle 1 = \angle 4$, and $\angle 3 = \angle 2$, why does OB bisect $\angle O$?



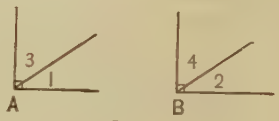
Exs. 6, 7

Ex. **8**. If $\angle 1 = \angle 2$, why does $\angle AOC = \angle BOD$?

Ex. **9**. If $\angle AOC = \angle BOD$, why does $\angle 1 = \angle 2$?

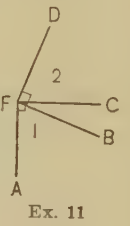
Exs. 8, 9

Ex. **10**. If $\angle A = \angle B$ and $\angle 1 = \angle 2$, why does $\angle 3 = \angle 4$?



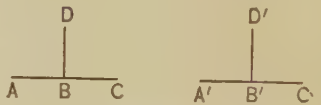
Ex. 10

Ex. **11**. If $\angle AFC = 90°$, and $\angle BFD = 90°$, why does $\angle 1 = \angle 2$?



Ex. 11

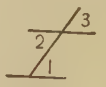
Ex. **12**. If st. $\angle ABC$ = st. $\angle A'B'C'$, why does rt. $\angle DBC$ = rt. $\angle D'B'C'$?



Ex. 12

Ex. **13**. If $\angle 1 + \angle b + \angle 2 = 180°$, $\angle 1 = \angle a$, and $\angle 2 = \angle c$, why does $\angle a + \angle b + \angle c = 180°$?

Ex. **14**. If $\angle 1 = \angle 2$, and $\angle 2 = \angle 3$, why does $\angle 1 = \angle 3$?



Ex. 14

BOOK I

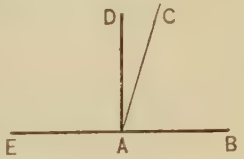
LINES AND RECTILINEAR FIGURES

PRELIMINARY THEOREMS

46. *All right angles are equal.*

For all straight angles are equal (Ax. 15), and halves of equals are equal (Ax. 8).

47. *At a given point in a given line one and only one perpendicular can be drawn to the line.*

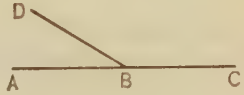


For if two perpendiculars, AD and AC, could be drawn at A, we should have two unequal right angles BAD and BAC, which is impossible (46).

48. *Complements of the same angle or of equal angles are equal.* (Ax. 3.)

49. *Supplements of the same angle or of equal angles are equal.* (Ax. 3.)

50. *If two adjacent angles have their exterior sides in a straight line, they are supplementary.*

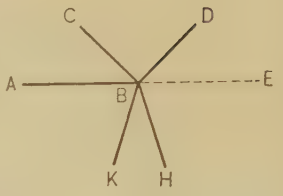


(Ax. 10.)

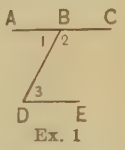
51. *If two adjacent angles are supplementary, their exterior sides are in the same straight line.*

Their exterior sides form a straight angle (36), and hence lie in a straight line (25).

52. *The sum of all the angles about a point in a plane is equal to two straight angles.*

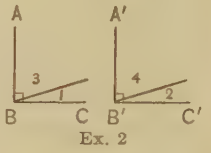


Ex. **1**. If ABC is a straight line, and $\angle 3$ is the supplement of $\angle 2$, why is $\angle 1 = \angle 3$?



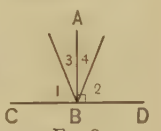
Ex. 1

Ex. **2**. If $AB \perp BC$, $A'B' \perp B'C'$, and $\angle 3 = \angle 4$, prove $\angle 1 = \angle 2$.



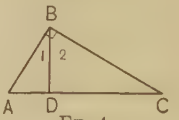
Ex. 2

Ex. **3**. If $AB \perp CD$ and $\angle 1 = \angle 2$, prove $\angle 3 = \angle 4$.



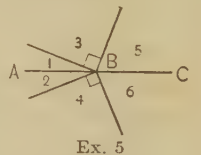
Ex. 3

Ex. **4**. If $\angle ABC$ is a right angle and $\angle A$ is the complement of $\angle 1$, prove $\angle A = \angle 2$.



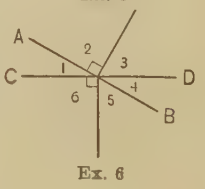
Ex. 4

Ex. **5**. If ABC is a st. $\angle$, $\angle 1 = \angle 2$, and $\measuredangle 3$ and 4 are rt. $\measuredangle$, prove that $\angle 5 = \angle 6$.



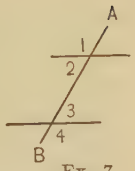
Ex. 5

Ex. **6**. If AB and CD are straight lines and $\measuredangle 2$ and 6 are right angles, prove that $\angle 3 = \angle 5$



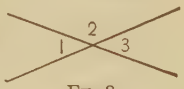
Ex. 6

Ex. 7. If AB is a straight line and $\angle 2 = \angle 3$, prove that $\angle 1 = \angle 4$.



Ex. 7

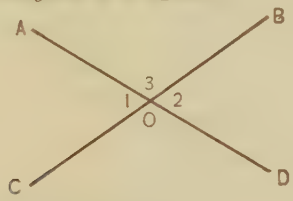
Ex. 8. If $\angle 1$ is the supplement of $\angle 2$ and $\angle 3$ is the supplement of $\angle 2$, why is $\angle 1 = \angle 3$?



Ex. 8

PROPOSITION I. THEOREM

53. *Vertical angles are equal.*



Given the vertical $\measuredangle$ 1 and 2.

To prove $\angle 1 = \angle 2$.

Proof

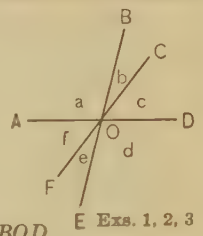
STATEMENTS	REASONS
1. AOD and COB are straight lines.	1. By hyp. $\angle 1$ and $\angle 2$ are vertical $\measuredangle$.
2. $\angle 1$ is the sup. of $\angle 3$.	2. If two adj. $\measuredangle$ have their ext. sides in a st. line, they are sup.
3. $\angle 2$ is the sup. of $\angle 3$.	3. Same reason as 2.
4. $\therefore \angle 1 = \angle 2$.	4. Supplements of the same $\angle$ are equal.

54. Every proof consists of a number of statements, each of which is supported by a definite reason. The only admissible reasons are: a previously proved proposition; an axiom; a definition; or the hypothesis.

[Practical applications, pages 281–286.]

Ex. 1. Three straight lines, AD, BE, and CF, meet in O, forming the six angles: a, b, c, d, e, and f.

(a) If $\angle b = 20°$, and $\angle c = 60°$, find $\angle AOE$.
(b) If $\angle FOB = 130°$, and $\angle c = 40°$, find $\angle a$.
(c) If $\angle FOD = 140°$, and $\angle b = 60°$, find $\angle a$.
(d) If $\angle f = 60°$, and $\angle b = 25°$, find $\angle d$.
(e) If $\angle b$ and f are complementary, find $\angle d$.
(f) If $\angle f = \angle b$, and $\angle d = 100°$, find $\angle b$.
(g) If $\angle a = 2\angle c$, and $\angle e = 60°$, find $\angle c$.
(h) If $\angle AOC = 140°$, and $\angle COE = 120°$, find $\angle BOD$.



Exs. 1, 2, 3

Ex. 2. In the same diagram prove that

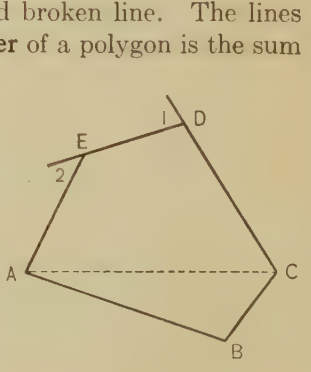
(a) $\angle b + \angle c = \angle AOE$.
(b) $\angle FOB - \angle c = \angle a$.
(c) $\angle FOB - \angle f = \angle d$.
(d) $\angle f + \angle b + \angle d = 180°$.
(e) $\angle AOC + \angle BOD + \angle COE = 360°$.
(f) $\angle AOC + \angle COE - \angle EOA = 2\angle a$.
(g) If $\angle f = \angle e$, then $\angle b = \angle c$.

[1]* Ex. 3. In the same diagram

(a) If $\angle AOC = 150°$, and $\angle COE = 130°$, find $\angle a$.
(b) If $\angle FOB = 140°$, and $\angle AOC = 125°$, find $\angle d$.
(c) If $\angle AOE + \angle BOC = 140°$, and $\angle c = 40°$, find $\angle e$.
(d) If $\angle FOB = \angle FOD$, prove that $\angle f = \angle e$.

55. DEF. A **polygon** is a closed broken line. The lines are called the **sides**. The **perimeter** of a polygon is the sum of all its sides.

The angles formed by the consecutive sides are the **angles** of the polygon, and their vertices are the **vertices** of the polygon. An **exterior angle** is formed by a side and the prolongation of a consecutive one. A **diagonal** is a straight line joining two vertices not on a same side of the polygon.



[1] Exercises denoted by (*) are difficult and may be omitted in a first reading.

Thus $ABCDE$ is a polygon of five sides, AB and BC are sides, $\angle A$ is an angle, point B is a vertex, and AC is a diagonal of the polygon. $\angle 1$ and $\angle 2$ are exterior angles of the polygon.

56. DEF. A **quadrilateral** is a polygon of four sides.

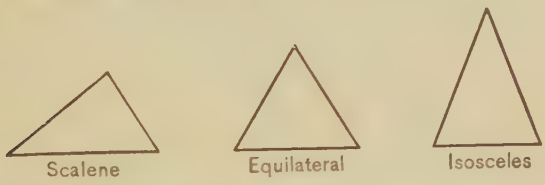
TRIANGLES — PART I

57. A **triangle** is a polygon of three sides.

58. Triangles classified as to sides. A **scalene** triangle is a triangle no two sides of which are equal.

An **isosceles** triangle is a triangle having two equal sides.

An **equilateral** triangle is a triangle having all of its sides equal. It is a special kind of isosceles triangle.

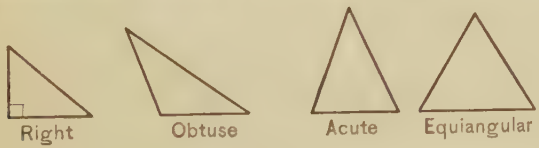


59. Triangles classified as to angles. A **right** triangle is a triangle in which one angle is *a right angle*.

An **obtuse** triangle is a triangle in which one angle is *an obtuse angle*.

An **acute** triangle is a triangle in which *all angles* are *acute*.

An **equiangular** triangle is a triangle in which *all angles* are equal.



60. DEF. The **base** of a triangle is the side on which the figure appears to stand. Any side may be considered as a base.

The two equal sides of an isosceles triangle are sometimes called the *arms* (or *legs*); the third side is called the *base*.

The angles opposite the arms are *the base angles* of an isosceles triangle.

61. DEF. The **vertex angle of a triangle** is the angle opposite the base.

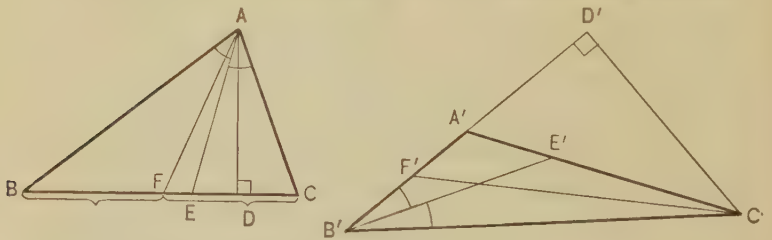
The vertex of this angle is sometimes called "the vertex of the triangle."

62. DEF. The **hypotenuse** of a right triangle is the side opposite the right angle.

The sides including the right angle are sometimes called the *arms* (*legs*) of the right triangle.

63. DEF. An **altitude** of a triangle is the perpendicular from any vertex to the opposite side (produced if necessary).

Thus, AD and $C'D'$ are altitudes. Every triangle has three altitudes.



64. DEF. A **median** of a triangle is the line drawn from any vertex to the mid-point of the opposite side.

Thus, AF and $C'F'$ are medians. Every triangle has three medians.

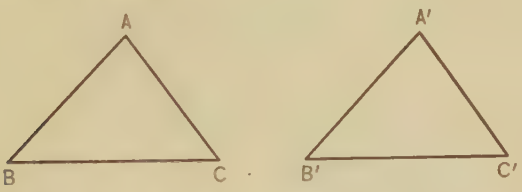
65. DEF. An **angle-bisector** of a triangle is the line drawn from any vertex bisecting the angle at that vertex and limited by the side opposite this vertex.

Thus, AE and $B'E'$ are angle-bisectors. Every triangle has three angle-bisectors.

NOTE. Students who are familiar with the exercises on geometric drawing (39) should construct exactly a number of the diagrams defined above, as right △, obtuse △, medians, angle-bisectors, equilateral △, etc.

Proposition II. Theorem

66. *Two triangles are congruent if two angles and the included side of one are equal respectively to two angles and the included side of the other* ($a.\,s.\,a. = a.\,s.\,a.$).



Given $\triangle ABC$ and $A'B'C'$ with $AB = A'B'$, $\angle A = \angle A'$, and $\angle B = \angle B'$.

To prove $\triangle ABC \cong \triangle A'B'C'$.

Proof

STATEMENTS	REASONS
1. Place $\triangle ABC$ upon $\triangle A'B'C'$ so that AB shall coincide with its equal $A'B'$.	**1.** Ax. 19. $AB = A'B'$, by hyp.
2. Then BC will fall along $B'C'$.	**2.** $\angle B = \angle B'$, by hyp.
3. AC will fall along $A'C'$.	**3.** $\angle A = \angle A'$, by hyp.
4. C will fall upon C'.	**4.** Two st. lines can intersect in only one point.
5. $\therefore \triangle ABC \cong \triangle A'B'C'$.	**5.** Figures are congruent if they can be made to coincide.

67. This method of proof (**superposition**) is employed in fundamental propositions only. The student should place some parts upon each other whose equality is known, and, by successive steps, trace the position of the rest of the figure.

Note. In order to facilitate the citing of propositions, the following abbreviation is suggested for the above proposition: a. s. a. = a. s. a.

68. Def. Polygons are **mutually equiangular** if their angles are respectively equal, and **mutually equilateral** if their sides are respectively equal.

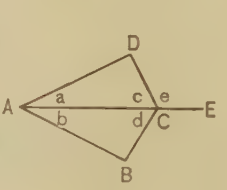
If two polygons are mutually equiangular, lines or angles similarly situated are called **corresponding** (or **homologous**) **lines or angles.** Thus AB and $A'B'$ (Prop. II) are corresponding sides; C and C', corresponding angles, the medians drawn from A and A' respectively, corresponding medians; etc.

Ex. 1. **Hyp.** $\angle a = 30°$, $\angle b = 30°$, $\angle c = 60°$, $\angle d = 60°$.

To prove $\triangle ABC \cong \triangle ADC$.

Ex. 2. **Hyp.** $\angle a = 40°$, $\angle b = 40°$, $\angle e = 130°$, $\angle d = 50°$, and ACE is a straight line.

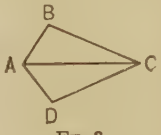
To prove $\triangle ABC \cong \triangle ADC$.



Exs. 1, 2

Ex. 3. **Hyp.** In quadrilateral $ABCD$, AC bisects angle A, and AC bisects $\angle C$.

To prove $\triangle ABC \cong \triangle ADC$.



Ex. 3

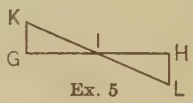
Ex. 4. **Hyp.** BD bisects $\angle B$, $EF \perp BD$.

To prove $\triangle EBD \cong \triangle FBD$.

Ex. 4

Ex. 5. **Hyp.** I is the mid-point of GH, $KG \perp GH$, $HL \perp GH$, KL is a straight line.

To prove $\triangle KGI \cong \triangle HLI$.



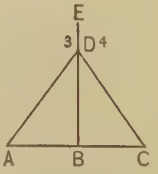
Ex. 5

Ex. 6. **Hyp.** $DB \perp AC$, $\angle 3 = \angle 4$, and BDE is a straight line.

To prove $\triangle ABD \cong \triangle CBD$.

Ex. 7. **Hyp.** $DB \perp AC$, $\angle ADB$ is the sup. of $\angle 4$, and BDE is a st. line.

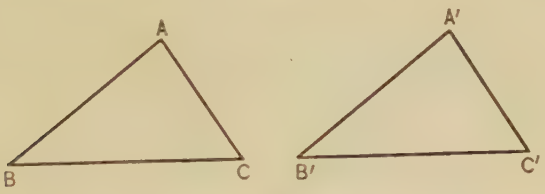
To prove $\triangle ABD \cong \triangle CBD$.



Exs. 6, 7

Proposition III. Theorem

69. *Two triangles are congruent if two sides and the included angle of one are equal respectively to two sides and the included angle of the other* (*s. a. s. = s. a. s.*)



Given $\triangle ABC$ and $A'B'C'$ with $AB = A'B'$, $BC = B'C'$, and $\angle B = \angle B'$.

To prove $\triangle ABC \cong \triangle A'B'C'$.

Proof

STATEMENTS	REASONS
1. Place $\triangle ABC$ upon $\triangle A'B'C'$ so that BC shall coincide with its equal $B'C'$.	**1.** A geom. fig. may be . . . (Ax. 19.); $BC = B'C'$, by hyp.
2. BA will fall along $B'A'$.	**2.** $\angle B = \angle B'$, by hyp.
3. The point A will fall upon the point A'.	**3.** $AB = A'B'$, by hyp.
4. AC will coincide with $A'C'$.	**4.** One st. line and only one can be drawn through two points.
5. $\therefore \triangle ABC \cong \triangle A'B'C'$.	**5.** Figures are congruent if they can be made to coincide.

Ex. 1. **Hyp.** Two lines AB and CD bisect each other in E.

To prove $\triangle ADE \cong \triangle CEB$.[1]

[1] The following set of exercises is of fundamental importance. No student should go on with the work until he is able to discover the proofs of all the simpler theorems given below. For additional work of this kind see Schultze's Teaching of Mathematics.

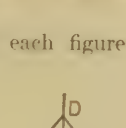
70. Method I. The equality of lines and of angles is usually proved by means of congruent triangles.

Note 1. The equal lines will be the corresponding sides in the congruent triangles. Likewise for the equal angles.

Note 2. The student is advised to draw freehand each figure accompanying each exercise.

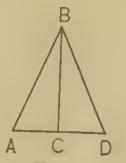
Ex. 2. **Hyp.** DC is the perpendicular bisector of AB, (*i.e.* $AC = CB$ and $DC \perp AB$.)

To prove $AD = BD$.



Ex. 2

Ex. 3. The bisector of the vertex angle of an isosceles triangle bisects the base.



Ex. 3

Ex. 4. **Hyp.** BF bisects $\angle DBE$.
$BD = BE$.

To prove $\angle FDB = \angle FEB$.

Ex. 4

Ex. 5. **Hyp.** $BD = BF$, $BC = BA$, and CG and AE are st. lines.

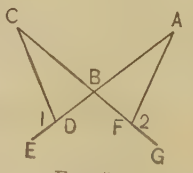
To prove $CD = AF$.

Ex. 6. **Hyp.** $BD = BF$, $CD \perp EA$, $AF \perp CG$, and CG and AE are st. lines.

To prove $\angle C = \angle A$.

Ex. 7. **Hyp.** $BD = BF$, $\angle 1 = \angle 2$, and CG and AE are st. lines.

To prove $CB = AB$.



Exs. 5–8

Ex. 8. **Hyp.** $CF = AD$, $DB = FB$, and CG and AE are st. lines.

To prove $\angle C = \angle A$.

Ex. 9. Hyp. $AD = DC$, $\angle 5 = \angle 6$, and BE is a st. line.

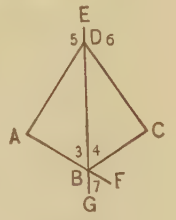
To prove $\angle A = \angle C$.

Ex. 10. Hyp. $\angle 3 = \angle 4$, $\angle 5 = \angle 6$, and BE is a st. line.

To prove $AD = DC$.

Ex. 11. Hyp. $AB = BC$, $\angle 4 = \angle 7$, and AF and EG are st. lines.

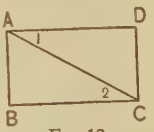
To prove $AD = DC$.



Exs. 9–11

Ex. 12. Hyp. $AB \perp AD$, $DC \perp BC$ and $\angle 1 = \angle 2$.

To prove $\angle B = \angle D$.



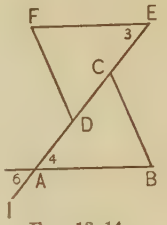
Ex. 12

Ex. 13. Hyp. $AD = CE$, $AB = FE$, $\angle 3 = \angle 4$, and AE is a st. line.

To prove $\angle B = \angle F$.

Ex. 14. Hyp. $AD = CE$, $AB = FE$, $\angle 3 = \angle 6$, and EI and HB are st. lines.

To prove $BC = FD$.



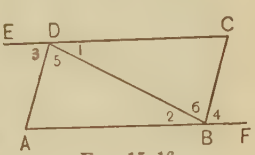
Exs. 13, 14

Ex. 15. Hyp. $\angle EDB = \angle FBD$, $\angle 5 = \angle 6$, and CE and AF are st. lines.

To prove $AD = BC$.

Ex. 16. Hyp. $\angle 3 = \angle 4$, $\angle 5 = \angle 6$, and EC and AF are st. lines.

To prove $AB = DC$.



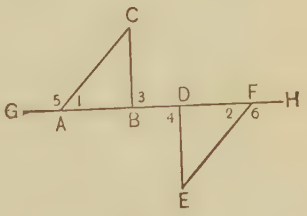
Exs. 15, 16

Ex. 17. Hyp. $AD = BF$, $\angle 1 = \angle 2$, $\angle 3 = \angle 4$, and GH is a st. line.

To prove $\angle C = \angle E$.

Ex. 18. Hyp. $AD = BF$, $\angle 5 = \angle 6$, $AC = FE$, and GH is a st. line.

To prove $BC = DE$.

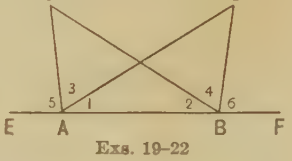


Exs. 17, 18

Ex. **19.** **Hyp.** $AD \perp EF, CB \perp EF, AD = BC$, and EF is a st. line.
To prove $\angle C = \angle D$.

Ex. **20.** **Hyp.** $AD = CB$, $\angle 5 = \angle 6$, and EF is a st. line.
To prove $\angle 1 = \angle 2$.

Ex. **21.** **Hyp.** $\angle 5 = \angle 6$, $\angle 1 = \angle 2$, and EF is a st. line.
To prove $AC = DB$.



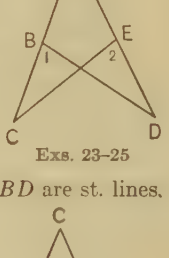
Exs. 19–22

Ex. **22.** **Hyp.** $\angle 5 = \angle 6$, $\angle 3 = \angle 4$, and EF is a st. line.
To prove $AD = BC$.

Ex. **23.** **Hyp.** $AB = AE, BC = ED$, and AC and AD are st. lines.
To prove $CE = BD$.

Ex. **24.** **Hyp.** $AB = AE$, $\angle 1 = \angle 2$, and AC and AD are st. lines.
To prove $CE = BD$.

[1] Ex. **25.** **Hyp.** $AC = AD$, and $BC = ED$.
To prove $\angle C = \angle D$.

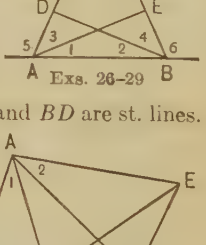


Exs. 23–25

Ex. **26.** **Hyp.** $\angle 1 = \angle 2$, $\angle 3 = \angle 4$, and AE and BD are st. lines.
To prove $AD = BE$.

Ex. **27.** **Hyp.** $\angle DAB = \angle EBA$, and $\angle 3 = \angle 4$.
To prove $AD = BE$.

Ex. **28.** **Hyp.** $\angle 5 = \angle 6$, $\angle 1 = \angle 2$, and AE and DB are st. lines.
To prove $AD = BE$.

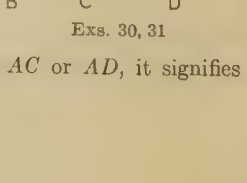


Exs. 26–29

Ex. **29.** **Hyp.** $\angle 5 = \angle 6$, $AD = BE$, and AE and BD are st. lines.
To prove $\angle 1 = \angle 2$.

Ex. **30.** **Hyp.** BCD is a st. line, $AB = AC$, $AD = AE$, and $\angle 1 = \angle 2$.
To prove $BD = CE$.

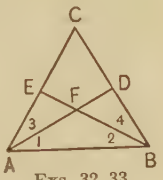
Ex. **31.** **Hyp.** $AB = AC$, $\angle 1 = \angle 2$, $\angle 3 = \angle 4$, and AD is a st. line.
To prove $AD = AE$.



Exs. 30, 31

[1] If a line is denoted by two letters only, as AC or AD, it signifies that it is a straight line.

NOTE. The following exercises are more difficult because the two parts whose equality is to be demonstrated may be considered corresponding parts of two different pairs of triangles, and the student has to determine by trial which pair must be used.

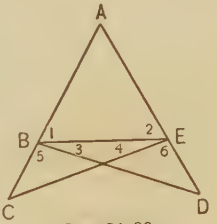


Exs. 32, 33

Ex. 32. Hyp. $\angle A = \angle B$, and $\angle 3 = \angle 4$.
To prove $AD = BE$.

Ex. 33. Hyp. $AC = BC$, and $\angle 3 = \angle 4$.
To prove $AD = BE$.

Ex. 34. Hyp. $\angle 1 = \angle 2$, and $\angle 3 = \angle 4$.
To prove $EC = BD$.



Exs. 34–36

***Ex. 35. Hyp.** $\angle 1 = \angle 2$, $DB \perp AC$, and $CE \perp AD$.
To prove $\angle C = \angle D$.

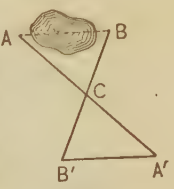
Ex. 36. Hyp. $AB = AE$, $\angle 5 = \angle 6$.
To prove $EC = BD$.

Ex. **37**. *If two angles of a triangle are equal, the corresponding angle-bisectors are equal.*

Ex. **38**. *The medians drawn to the arms of an isosceles triangle are equal.*

Ex. **39**. If two exterior angles of a triangle are equal, the angle-bisectors of the adjacent interior angles are equal.

Ex. **40**. Wishing to determine the distance across a pond, AB, we place a stake at a convenient station C, and by sighting from A, we locate A' in the prolongation of AC, and make $CA' = AC$. Similarly we produce BC to B' so that $CB' = BC$. Which line must we measure to obtain AB? Prove your statement.



Ex. 40

PROPOSITION IV. THEOREM

71. *The base angles of an isosceles triangle are equal.*

Given $\triangle ABC$ with $AC = BC$.

To prove $\angle A = \angle B$.

Proof

STATEMENTS	REASONS
1. Draw the bisector of $\angle C$ and extend it to meet AB at D.	**1.** Every $\angle$ has a bisector. A line may be extended.
2. In $\triangle ACD$ and BCD, $AC = CB$.	**2.** Hyp.
3. $\angle 1 = \angle 2$.	**3.** By construction.
4. $CD = CD$.	**4.** Iden.
5. $\triangle ACD \cong \triangle BCD$.	**5.** Two $\triangle$ are $\cong$ if two sides and the included angle of one are equal respectively to two sides and the included angle in the other.
6. $\therefore \angle A = \angle B$.	**6.** Corresponding angles of congruent $\triangle$ are equal.

72. COR. 1. *The bisector of the vertex angle of an isosceles triangle is perpendicular to the base and bisects the base.*

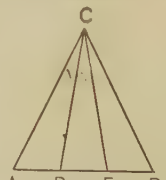
73. COR. 2. *An equilateral triangle is equiangular.*

Ex. **1.** If $AC = BC$, and $AE = DB$, then $CD = CE$.

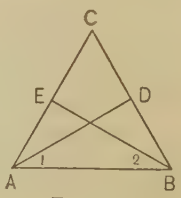
Ex. **2.** If $AC = BC$, and $\angle DCA = \angle ECB$, then $CD = CE$.

Ex. **3.** If $AC = BC$, and $\angle 1 = \angle 2$, then $AE = BD$.

Ex. **4.** If $AC = BC$, and AD and BE are angle-bisectors, then $AD = BE$.



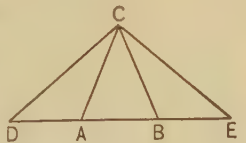
Exs. 1, 2



Exs. 3, 4

Ex. 5. If $AC = BC$, $AD = BE$, and DE is a straight line, then DEC is an isosceles triangle.

Ex. 6. If $AC = BC$, $\angle ACD = \angle BCE$, and DE is a straight line then DEC is an isosceles triangle.

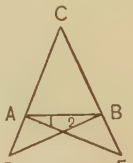


Exs. 5, 6

*Ex. 7. If $AC = BC$, $\angle 1 = \angle 2$, and CD and CE are straight lines, then $\angle D = \angle E$.

Ex. 8. If the base of an isosceles triangle is trisected, the lines joining the points of division with the vertex are equal.

Ex. 9. If in the sides of an equilateral $\triangle ABC$, the points D, E, and F be taken so that $AD = BE = CF$, then $\triangle DFE$ is equilateral.

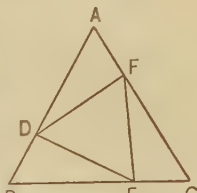


Ex. 7

Ex. 10. Lines drawn from the mid-points of the arms of an isosceles triangle to the mid-point of the base are equal.

Ex. 11. If $DE = EF = FD$, and $\angle AFD = \angle BDE = \angle CEF$, then $\triangle ABC$ is equilateral.

Note. The equality of angles is sometimes proved by means of Prop. IV (compare § 123).



Exs. 9, 11

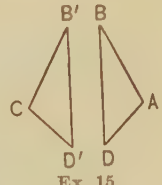
Ex. 12. If ABC and ADC are two isosceles triangles on the same base, AC, then $\angle BAD = \angle BCD$.

Ex. 13. If in quadrilateral $ABCD$, $AB = BC$ and $AD = DC$, then $\angle A = \angle C$.

Ex. 12

Ex. 13

Ex. 14



Ex. 15

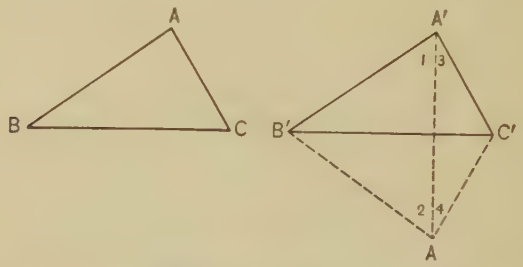
Ex. 14. If two triangles ABD and DBC have BD common, $AB = BC$ and $AD = DC$, then $\angle A = \angle C$.

Ex. 15. If in two triangles ABD and $CB'D'$, $AD = CD'$, $AB = CB'$, and $BD = B'D'$, then $\angle A = \angle C$.

Hint. Place the $\triangle$ together, so as to form a quadrilateral.

Proposition V. Theorem

74. *Two triangles are congruent if three sides of one are equal respectively to three sides of the other* (*s. s. s.* = *s. s. s.*).



Given $\triangle ABC$ and $A'B'C'$, with $AB = A'B'$, $BC = B'C'$, $AC = A'C'$.

To prove $\triangle ABC \cong \triangle A'B'C'$.

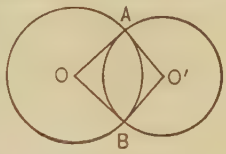
Proof

STATEMENTS	REASONS
1. Place $\triangle ABC$ so that BC shall coincide with $B'C'$ and A and A' lie on opposite sides of $B'C'$.	**1.** Axiom 19. $BC = B'C'$, by hyp.
2. Draw AA'.	**2.** A st. line may be drawn between two points.
3. $\triangle AB'A'$ is isosceles.	**3.** $AB = A'B'$ by hyp.
4. $\therefore \angle 1 = \angle 2$.	**4.** The base $\measuredangle$ of an isos. $\triangle$ are equal.
5. $\triangle AC'A'$ is isosceles.	**5.** $AC = A'C'$ by hyp.
6. $\therefore \angle 3 = \angle 4$.	**6.** Same as 4.
7. $\therefore \angle 1 + \angle 3 = \angle 2 + \angle 4$.	**7.** If equals are added to equals, the sums are equal.
8. Or $\angle A = \angle A'$.	**8.** Sub.
9. $\therefore \triangle AB'C' \cong \triangle A'B'C$. *i.e.* $\triangle ABC \cong \triangle A'B'C'$.	**9.** s. a. s. = s. a. s.

Ex. 1. *The median to the base of an isosceles triangle bisects the vertex angle.*

Ex. 2. *If the opposite sides of a quadrilateral are equal, the opposite angles are equal.*

75. METHOD II. **If the lines or angles whose equality we wish to demonstrate are not parts of congruent triangles, we try to make them parts of congruent triangles by drawing additional lines.**

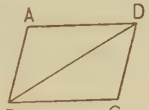


Exs. 1, 2

Ex. 1. If two circles whose centers are O and O' intersect in A and B, then $\angle AOO' = \angle BOO'$.

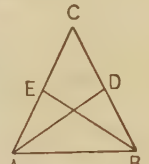
Ex. 2. If two circles whose centers are O and O' intersect in A and B, then $\angle OAO' = \angle OBO'$.

Ex. 3. If $AB = CD$ and $BC = DA$, then $\angle ABD = \angle BDC$.



Ex. 3

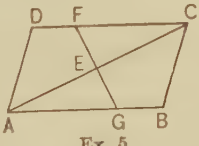
Ex. 4. If $AE = BD$, $AD = BE$, and AC and BC are st. lines, then $\angle CEB = \angle CDA$.



Ex. 4

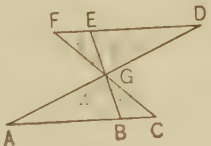
76. METHOD III. **If it is impossible to prove the congruence of the required pair of triangles, prove first the congruence of some other pair, or pairs, whose corresponding parts will enable us to demonstrate the congruence of the original pair.**

Ex. 5. If the opposite sides of a quadrilateral are equal, and a line be drawn through the mid-point of a diagonal, terminating in two sides, this line is bisected.



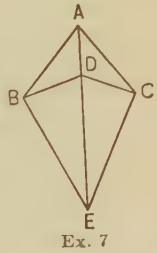
Ex. 5

Ex. 6. If $AG = GD$, $CG = FG$, and all lines are straight lines, then $BG = GE$.



Ex. 6

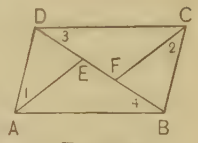
Ex. 7. If $AB = AC$, AE bisects $\angle BAC$, and AE is a st. line, then $\angle DBE = \angle DCE$.



Ex. 7

Ex. 8. If $AB = CD$, $BC = DA$, and $\angle 1 = \angle 2$, and DB is a st. line, then $AE = CF$.

Ex. 9. If $AB = CD$, $\angle 3 = \angle 4$, and $BF = DE$, then $\angle 1 = \angle 2$.



Exs. 8, 9

Ex. 10. Two triangles are congruent if two sides and the median to one of the sides are equal respectively to two sides and the corresponding median of the other.

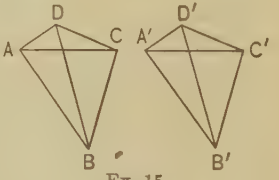
Ex. 11. Two isosceles triangles are congruent if an arm and the line which joins the mid-points of the arms of one triangle are equal respectively to an arm and the line which joins the mid-points of the arms of the other.

Ex. 12. *If the opposite sides of a quadrilateral are equal, the diagonals bisect each other.*

Ex. 13. Two triangles ABC and $A'B'C'$ are congruent if $AB = A'B'$, $\angle A = \angle A'$, and angle-bisector AD = angle-bisector $A'D'$.

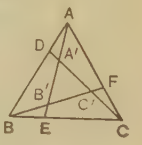
Ex. 14. If in quadrilateral $ABCD$, $AB = BC$, $CD = DA$, and diagonals AC and BD meet in E, then $AE = EC$.

Ex. 15. If in quadrilaterals $ABCD$ and $A'B'C'D'$, $AB = A'B'$, $BC = B'C'$, $CD = C'D'$, $DA = D'A'$, and $AC = A'C'$, then $BD = B'D'$.



Ex. 15

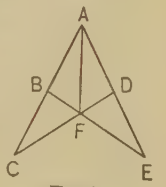
* Ex. 16. If on the sides of an equilateral triangle ABC, the points D, E, F are taken, so that $AD = BE = CF$, and E, D, and F are joined to the opposite vertices, then $\triangle A'B'C'$ is equilateral.



Ex. 16

* Ex. 17. If the opposite sides of a polygon of six sides are equal, and two of the opposite angles are equal, then all opposite angles are equal.

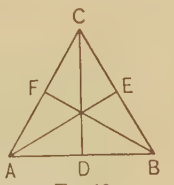
* Ex. 18. If $AB = AD$, $AC = AE$, and BE and DC are st. lines, then $\angle BAF = \angle DAF$.



Ex. 18

* See footnote on page 20.

*Ex. 19. If $\angle A = \angle B$, and $AF = BE$, then $AD = DB$.



Ex. 19

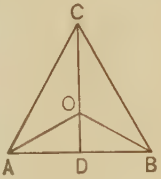
77. METHOD IV. **To prove that an angle is a right angle we usually demonstrate that it is equal to its supplementary adjacent angle.**

Ex. **20.** *The diagonals of an equilateral quadrilateral are perpendicular to each other.*

Ex. **21.** *The median to the base of an isosceles triangle is perpendicular to the base.*

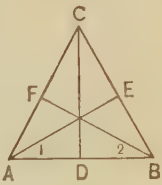
Ex. **22.** In quadrilateral $ABCD$, if $AB = BC$, and $CD = DA$, then $BD \perp AC$.

Ex. **23.** If $AC = BC$, and $AO = BO$, then $CD \perp AB$.



Ex. 23

*Ex. **24.** If $AC = BC$,
and $\angle 1 = \angle 2$,
then $CD \perp AB$.



Ex. 24

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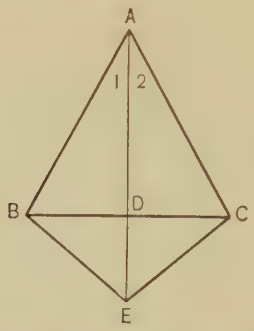
78. A point C is said to be **equidistant from two other points A and B if $AC = BC$.**

It is not necessary that these two lines be drawn. Thus, we may say D is equidistant from A and B, even if DA and DB are not drawn. In general, two letters, as AD, mean the straight line connecting A and D.

Proposition VI. Theorem

79. *Two points each equidistant from the ends of a line, determine the perpendicular bisector of the line.*



Given line BC and points A and E such that $AB = AC$, $BE = EC$.

To prove AE is the perpendicular bisector of BC.

Proof

STATEMENTS	REASONS
1. $AB = AC$, $BE = EC$.	**1.** Hyp.
2. $AE = AE$.	**2.** Iden.
3. $\triangle ABE \cong \triangle AEC$.	**3.** s. s. s. = s. s. s.
4. $\angle 1 = \angle 2$.	**4.** Corr. $\measuredangle$ of $\cong$ $\triangle$ are $=$.
5. $\triangle ABC$ is isosceles.	**5.** $AB = AC$, by hyp.
6. St. line ADE is the $\perp$ bisector of BC.	**6.** The bisector of the vertex $\angle$ of an isos. $\triangle$ is $\perp$ to the base and bisects the base.

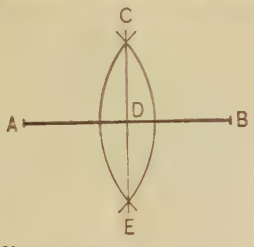
80. Cor. *Every point equidistant from the ends of a line lies in the perpendicular bisector of that line.*

Ex. *If the four sides of a quadrilateral are equal, the diagonals bisect each other.*

CONSTRUCTIONS

Proposition VII. Problem

81. *To bisect a given straight line.*



Given a straight line AB.

Required to bisect AB.

Construction. From A and B as centers, with equal radii greater than $\frac{1}{2}\,AB$, describe arcs intersecting at C and E.

Draw CE.

Then the line CE bisects AB at D.

Proof

STATEMENTS	REASONS
1. C is equidistant from A and B.	**1.** Con.
2. E is equidistant from A and B.	**2.** Con.
3. $\therefore AD = DB$.	**3.** Two points each equidistant from the ends of a line determine the perpendicular-bisector to that line.

82. Cor. By means of the preceding construction we can construct the perpendicular bisector (CE) of a given line (AB).

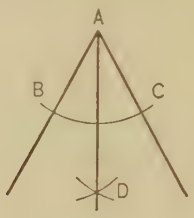
Ex. **1.** *Divide a given line into four equal parts.*

Ex. **2.** *Construct the three medians of a triangle.*

Ex. **3.** *Construct the three perpendicular bisectors of the three sides of a triangle.*

PROPOSITION VIII. PROBLEM

83. *To bisect a given angle.*



Given $\angle CAB$.

Required to bisect $\angle CAB$.

Construction. With A as a center and any radius, as AB, describe an arc cutting the sides of the $\angle A$ at B and C.

From B and C as centers, with equal radii greater than one half the distance from B to C, describe two arcs intersecting at D. Draw AD.

AD is the required bisector.

HINT. Prove AD is the bisector by first drawing BD and DC and then proving two $\triangle$ are congruent.

84. It is advisable to draw the lines of construction either very thin or dotted, while the given and required lines are represented by heavy full lines. In complex constructions the resulting lines may be drawn in red or in blue.

Ex. **1.** *To divide a given angle into four equal parts.*

Ex. **2.** *To bisect a straight angle.*

Ex. **3.** *Construct an angle of 90°, of 45°.*

Ex. **4.** Construct an angle of 22° 30′.

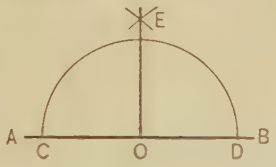
Ex. **5.** Construct an angle of 67° 30′.

Ex. **6.** Construct one half the supplement of a given angle.

Ex. **7.** *Construct the angle-bisectors of a given triangle.*

Proposition IX. Problem

85. *At a given point in a given straight line, to erect a perpendicular to that line.*



Given point O in line AB.

Required the perpendicular to the line AB at O.

Construction. From O as a center with any radius OC, describe an arc intersecting AB in C and D.

From C and D as centers with equal radii greater than OC, describe two arcs intersecting at E.

Draw OE.

OE is the required perpendicular.

Note 1. Draw CE and DE. Prove two $\triangle$ are congruent. Then show that OE is perpendicular to AB at O. See prop. 7, p. 37 for model for proof.

Note 2. Whenever a pupil has carried out a construction, he should prove the correctness of his construction.

Ex. 1. *Construct the complement of a given acute angle.*

Ex. 2. Construct the supplement of the complement of a given acute angle.

Ex. 3. Construct one half the complement of a given acute angle.

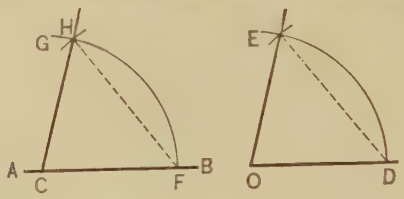
Ex. 4. Construct a right triangle given the arms equal to 1″ and $1\frac{1}{2}$″ respectively.

Ex. 5. Construct a square having a side equal to $1\frac{1}{8}$″.

Ex. 6. Construct a square given its perimeter equal to a given line.

Proposition X. Problem

86. *Through a given point on a straight line to draw a line making a given angle with the given line.*



Given point C in line AB and $\angle O$.

Required an $\angle$ at $C = \angle O$.

Construction. From O as a center with any radius, describe an arc cutting the sides of $\angle O$ in D and E.

From C as a center, with the same radius, describe arc FG, intersecting CB in F. From F as a center, with a radius equal to DE, draw an arc intersecting arc FG in H. Draw CH.

$\angle HCF$ is the required angle.

Hint. Prove this proposition by showing the two $\triangle$ are congruent.

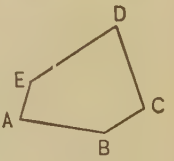
Note. Two letters, *e.g.* DE, used as above, denote a straight line.

Ex. 1. *Construct an angle equal to twice a given angle.*

Ex. 2. Construct an angle equal to twice the supplement of a given angle.

Ex. 3. Given a triangle ABC. Construct another triangle $A'B'C'$, having $A'B' = AB$, $\angle A' = \angle A$, and $\angle B' = \angle B$. What is the relation of $\triangle ABC$ and $A'B'C'$?

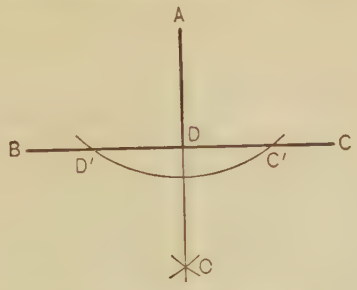
Ex. 4. Given a polygon of five sides (pentagon) $ABCDE$. Construct a pentagon having four of its sides equal respectively to AB, BC, CD, and DE, and the included angles respectively equal to $\measuredangle B$, C, and D.



Ex. 4

Proposition XI. Problem

87. *From a point without a straight line, to let fall a perpendicular upon that line.*



Given a straight line BC, and a point A without the line.

Required a perpendicular from the point A to the line BC.

Construction. From A as a center, with a radius sufficiently great, describe an arc cutting BC in C' and D'.

From D' and C' as centers, with equal radii greater than $\frac{1}{2}\ D'C'$, describe two arcs intersecting at O.

Draw AO intersecting BC in D.

AD is the required perpendicular.

[The proof is left to the student. Use § 79.]

Note. We can infer from this proposition that *from a point without a line one perpendicular can be drawn to the given line.*

Ex. 1. *Construct the three altitudes of an acute triangle.*

Ex. 2. *Construct the three altitudes of an obtuse triangle; of a right triangle.*

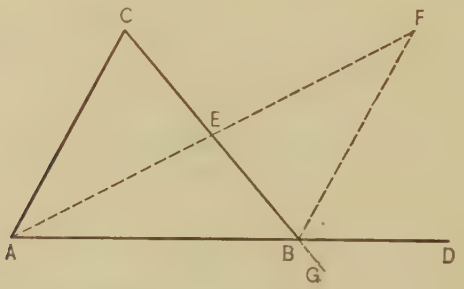
Ex. 3. From your construction of the three altitudes of a triangle, what can you say about these three lines?

Ex. 4. About what other lines in a triangle can you say the same thing?

Ex. 5. Construct a right triangle given hypotenuse and an angle of 45°.

Proposition XII. Theorem

88. *An exterior angle of a triangle is greater than either remote interior angle.*



Given $\triangle ABC$ and the ext. $\angle CBD$.

To prove $\angle CBD > \angle C$ or $\angle A$.

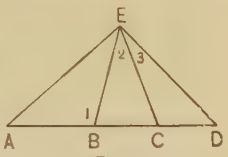
Proof

STATEMENTS	REASONS
1. Let E be the mid-point of BC. Draw AE and produce it its own length to F. Draw FB.	1. Every line has a bisector. A st. line can be drawn between two points. A line may be extended.
2. In $\triangle ACE$ and FBE, $AE = EF$.	2. Con.
3. $EC = BE$.	3. Con.
4. $\angle CEA = \angle FEB$.	4. Vertical $\measuredangle$ are equal.
5. $\triangle ACE \cong \triangle FBE$.	5. s. a. s. = s. a. s.
6. $\angle EBF = \angle C$.	6. Corr. parts of $\cong$ $\triangle$ are =.
7. $\angle CBD > \angle EBF$.	7. The whole is greater than any of its parts.
8. Hence $\angle CBD > \angle C$.	8. Sub.

Note. By joining the mid-point of AB to C, etc., it follows that

$$\angle ABG > \angle A, \quad \angle ABG = \angle CBD,$$
$$\therefore \angle CBD > \angle A.$$

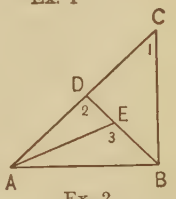
Ex. 1. If AD is a st. line, prove that (*a*) $\angle 1 > \angle 2$, (*b*) $\angle 1 > \angle D$, (*c*) $\angle 1 > \angle 3$.



Ex. 1

Ex. 2. If DB is a st. line, prove that (*a*) $\angle 2 > \angle 1$, (*b*) $\angle 3 > \angle 1$.

Ex. 3. In the diagram for Proposition XII, prove: (*a*) $\angle FBD > \angle F$, (*b*) $\angle BEA > \angle ACE$, (*c*) $\angle CEA > \angle GBD$, (*d*) $\angle CBD > \angle CEF$.



Ex. 2

89. DEF. A **transversal** is a line that intersects two (or more) other lines. The various angles formed by two lines and a transversal are named as follows:

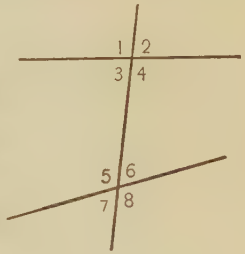
1, 2, 7, 8 are **exterior** angles.

3, 4, 5, 6 are **interior** angles.

1 and 8, 2 and 7 are **alternate exterior** angles.

3 and 6, 4 and 5 are **alternate interior** angles.

1 and 5, 2 and 6, 3 and 7, 4 and 8 are **corresponding** angles.



PARALLEL LINES

90. DEF. Two lines are **parallel** if they lie in the same plane and do not meet, however far produced (in each direction).

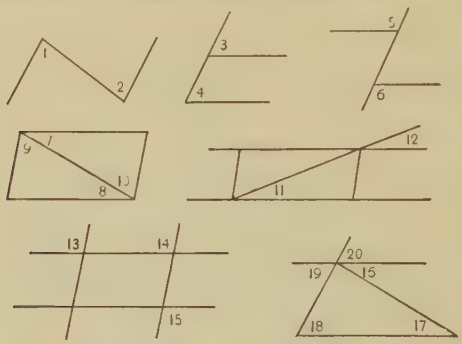
Thus, AB and CD represent two parallel lines.

91. AXIOM 16. *Two intersecting lines cannot both be parallel to a third straight line.*

NOTE. We assume that *two unlimited straight lines in the same plane either intersect or are parallel.*

92. THEOREM. *Two straight lines which are parallel to a third straight line are parallel to each other.*

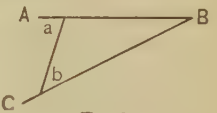
For if the two lines should meet, we should have two intersecting straight lines parallel to a third straight line, which contradicts Axiom 16.



Ex. **1**. In the diagrams given above, tell what kind of angles the following are: 1 and 2, 3 and 4, 5 and 6, 7 and 8, 9 and 10, 11 and 12, 13 and 14, 14 and 15, 16 and 17, 18 and 19, 18 and 20.

Ex. **2**. If AB and BC are st. lines, is it possible that

(*a*) $\angle a = 60°$, and $\angle b = 50°$?
(*b*) $\angle a = 60°$, and $\angle b = 70°$?
(*c*) $\angle a = 60°$, and $\angle b = 60°$?



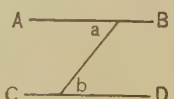
Ex. 2

Ex. **3**. Is it possible that the prolongations of AB and CD (*i.e.* toward the right) meet if

(*a*) $\angle a = 60°$, and $\angle b = 50°$?
(*b*) $\angle a = 60°$, and $\angle b = 70°$?
(*c*) $\angle a = 60°$, and $\angle b = 60°$?

Ex. **4**. Do the prolongations of BA and DC (*i.e.* toward the left) meet if

(*a*) $\angle a = 60°$, $\angle b = 70°$?
(*b*) $\angle a = 60°$, $\angle b = 50°$?
(*c*) $\angle a = 60°$, $\angle b = 60°$?



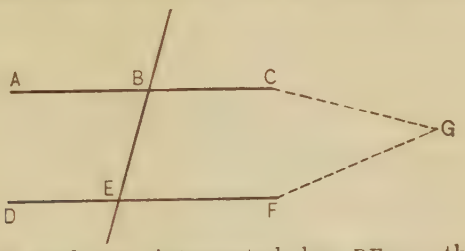
Exs. 3–5

Ex. **5**. Can the lines produced meet at all if

(*a*) $\angle a = 50°$, and $\angle b = 50°$?
(*b*) $\angle a = 60°$, and $\angle b = 60°$?

Proposition XIII. Theorem

93. *Two lines are parallel if a transversal to these lines makes a pair of alternate interior angles equal.*[1]



Given AC and DF, intersected by BE so that $\angle ABE = \angle BEF$.

To prove $AC \parallel DF$.

Proof

STATEMENTS	REASONS
1. $AC \parallel DF$ or AC not $\parallel DF$.	1. Only two suppositions are admissible. Sect. 91, Note.
2. If AC and DF are not parallel, they meet at a pt. G, forming $\triangle EBG$.	2. Def. of nonparallel lines.
3. $\angle ABE > \angle BEG$.	3. Ext. $\angle$ of a $\triangle >$ either remote interior angle.
4. This is impossible.	4. By hyp., $\angle ABE = \angle BEG$.
5. $\therefore AC \parallel DF$.	5. The other supposition has been proved false.

94. Method V. **To demonstrate that two lines are parallel, prove the equality of a pair of alternate interior angles.**

[1] In this proof we make a wrong assumption, viz., that two lines meet if the alt. int. $\measuredangle$ are equal. The diagram which illustrates this wrong assumption must necessarily be inaccurate, and hence the two angles are not exactly equal. Similarly, all proofs which require the investigation of a wrong assumption (the so-called indirect proofs) are illustrated by inaccurate drawings, and it is illogical to attempt to make these drawings exact.

Ex. 1. Prove that AB and CD are parallel if

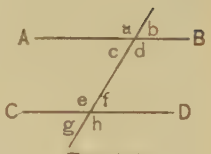
(a) $\angle c = 70°$, $\angle f = 70°$.

(b) $\angle c = 60°$, $\angle e = 120°$.

(c) $\angle a = 110°$, $\angle f = 70°$.

(d) $\angle b = 60°$, $\angle f = 60°$.

(e) $\angle a = 120°$, $\angle g = 60°$.



Exs. 1–4

Ex. 2. Prove that $AB \parallel CD$, if $\angle b = \angle f$.

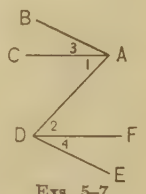
Ex. 3. Prove that $AB \parallel CD$, if $\angle a = \angle h$.

Ex. 4. Prove that $AB \parallel CD$, if $\angle b = \angle g$.

Ex. 5. If $\angle 1 = \angle 2$, and $\angle 3 = \angle 4$, prove that $AB \parallel DE$.

Ex. 6. If $\angle A = \angle D$, and $\angle 3 = \angle 4$, prove that $AC \parallel DF$.

Ex. 7. If $BA \perp AD$, $ED \perp AD$, and $\angle 3 = \angle 4$, prove that $AC \parallel DF$.

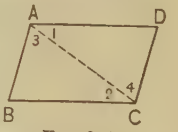


Exs. 5–7

Ex. 8. If AB and CD bisect each other in E, prove that $AC \parallel DB$.

Ex. 9. If $AB = DC$, and $\angle 3 = \angle 4$, prove that $AD \parallel BC$.

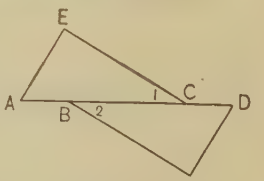
Ex. 10. If $AB = DC$, and $AD = BC$, prove that $AB \parallel DC$.



Exs. 9, 10

Ex. 11. If $AB = CD$, $EC = BF$, $\angle 1 = \angle 2$, and AD is a st. line, then $AE \parallel DF$.

Ex. 12. If $AE = DF$, $AB = CD$, and $EC = BF$, then $EC \parallel BF$.

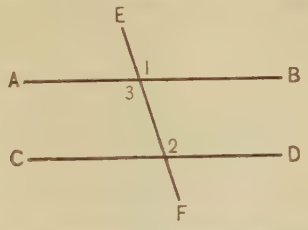


Exs. 11–13

Ex. 13. If $AB = CD$, $\angle ABF = \angle DCE$, $EC = BF$, and AD is a straight line, then $AE \parallel DF$.

PROPOSITION XIV. THEOREM

95. *Two lines are parallel if a transversal to these lines makes a pair of corresponding angles equal.*



Given AB and CD, intersected by EF so that $\angle 1 = \angle 2$.

To prove $AB \parallel CD$.

Proof

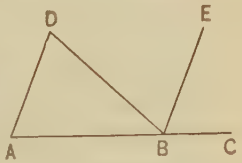
STATEMENTS		REASONS
1.	$\angle 1 = \angle 2$.	**1.** Hyp.
2.	$\angle 1 = \angle 3$.	**2.** Vert. $\measuredangle$ are $=$.
3.	$\angle 2 = \angle 3$.	**3.** Things $=$ to the same thing are $=$ to each other.
4.	$\therefore AB \parallel CD$.	**4.** Two lines are $\parallel$ if a pair of alt. int. $\measuredangle$ are $=$.

96. COR. *If two lines are perpendicular to a third line, they are parallel.*

Ex. 1. In the diagram for Ex. 1, p. 46, if $\angle a = 2\ \angle b$, and $\angle f = 60°$, then $AB \parallel CD$.

Ex. 2. If $\angle A = 70°$, $\angle ABD = 60°$, $\angle DBE = 50°$, and AC is a straight line, prove that $AD \parallel BE$.

Ex. 3. If BE bisects $\angle DBC$, $\angle ABD = 30°$, $\angle A = 75°$, and AC is a straight line, then $AD \parallel BE$.

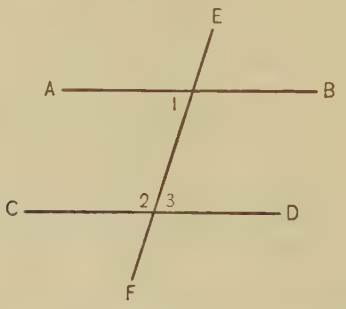


Exs. 2–4

Ex. 4. If $AB = BD$ and $\angle D = \angle EBC$, then $AD \parallel BE$.

Proposition XV. Theorem

97. *Two lines are parallel if a transversal to these lines makes a pair of interior angles on the same side of the transversal supplementary.*



Given AB and CD, intersected by EF so that $\angle 1$ and $\angle 2$ are supplementary.

To prove $AB \parallel CD$.

Proof

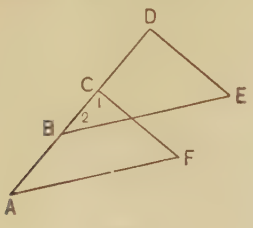
STATEMENTS	REASONS
1. $\angle 1$ is the sup. of $\angle 2$.	**1.** Hyp.
2. $\angle 3$ is the sup. of $\angle 2$.	**2.** If two adj. $\measuredangle$ have their ext. sides in a st. line, they are sup.
3. $\angle 1 = \angle 3$.	**3.** Supplements of the same $\angle$ are equal.
4. $\therefore AB \parallel CD$.	**4.** Lines are $\parallel$ if a pair of alt. int. $\measuredangle$ are equal.

98. Method VI. **Sometimes lines are demonstrated to be parallel by proving that:**

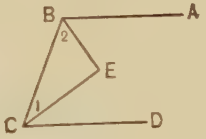
(*a*) **Two corresponding angles are equal, or**

(*b*) **Two interior angles on the same side of a transversal are supplementary.**

Ex. 1. If $AF = BE$, $AB = CD$, $\angle A = \angle 2$, and AD is a st. line, prove that $CF \parallel DE$.



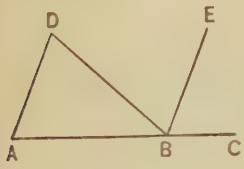
Ex. 1



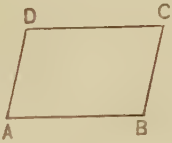
Ex. 2

Ex. 2. If BE bisects $\angle B$, CE bisects $\angle C$, and $\angle 1 + \angle 2 = 90°$, then $BA \parallel CD$.

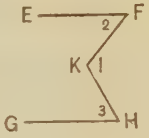
Ex. 3. If $AB = BD$ and $\angle D$ is the supplement of $\angle ABE$, then $AD \parallel BE$.



Ex. 3



Ex. 4



Ex. 5

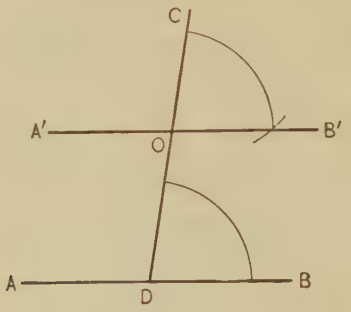
Ex. 4. If $\angle A + \angle B + \angle C + \angle D = 360°$, $\angle A = \angle C$, and $\angle B = \angle D$, then $AD \parallel BC$.

* Ex. 5. If $\angle 1 = \angle 2 + \angle 3$, then $EF \parallel GH$

99. To **construct figures** whose parts bear certain relations to each other, we employ the same methods as for proving these relations. Thus, to draw a line parallel to another, we may construct a pair of equal alt. int. $\measuredangle$; to make an $\angle$ equal to another $\angle$ we use a construction (86) which really involves the construction of congruent $\triangle$, etc.

Proposition XVI. Problem

100. *Through a given point to draw a parallel to a given line.*



Given line AB and point O.

Required the line through $O \parallel AB$.

Construction. Through O draw DC, intersecting AB in D. At O construct $\angle COB' = \angle ODB$. (86)
$A'B$ is the required line.

Hint. What are the means of proving that two lines are parallel?

Note. If previous constructions are employed in a problem, the details of such construction should *not* be stated. Thus, in the preceding problem it would be awkward and illogical to describe the method of making $\angle COB' = \angle ODB$.

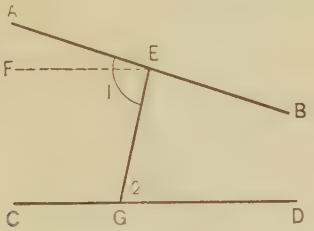
The beginner, however, should *draw* all such details. In other words the construction should be carried out exactly by means of ruler and compasses.

Ex. 1. Through a given point to draw a line parallel to a given line, by means of equal alt. int. $\measuredangle$.

Ex. 2. Through a given point to draw a line parallel to a given line by means of Ex. 8, p. 46.

Proposition XVII. Theorem

101. *Two lines are not parallel if a transversal to these lines makes a pair of alternate interior angles unequal.*



Given AB and CD cut by a transversal, so that

$$\angle 1 > \angle 2.$$

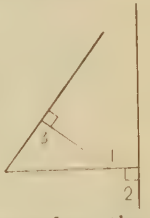
To prove AB is not $\parallel CD$.

Proof

STATEMENTS	REASONS
1. Draw EF so that $\angle FEG = \angle 2$.	1. To draw a line to make a given $\angle$ with a given line.
2. $EF \parallel CD$.	2. Two lines are $\parallel$ if a transversal makes the alt. int. $\measuredangle$ equal.
3. $\therefore$ AB is not $\parallel CD$.	3. Two intersecting lines cannot both be $\parallel$ to a third line.

102. Cor. *Two lines respectively perpendicular to two intersecting lines are not parallel.*

Hint. Prove $\angle 1 > \angle 2$.

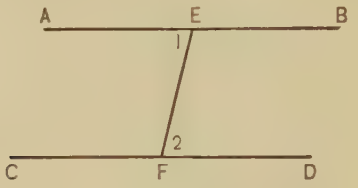


103. A proposition is the **converse** of another when the hypothesis and the conclusion of the one are respectively the conclusion and the hypothesis of the other.

Proposition XVIII. Theorem

104. *If two parallels are cut by a transversal, the alternate interior angles are equal.*

[Converse of Prop. XIII.]



Given parallel lines AB and CD, and the alt. int. $\measuredangle$ 1 and 2.

To prove $\angle 1 = \angle 2$.

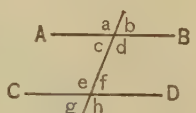
Proof

STATEMENTS	REASONS
1. Either $\angle 1 = \angle 2$, or $\angle 1 \neq \angle 2$.	1. Only two suppositions are admissible.
2. Suppose that $\angle 1 \neq \angle 2$, then AB is not $\parallel$ to CD.	2. Two lines are not $\parallel$ if a transversal to these lines makes a pair of alt. int. $\measuredangle$ unequal.
3. But this is impossible.	3. $AB \parallel CD$, by hyp.
4. $\therefore \angle 1 = \angle 2$.	4. The other supposition has been proved false.

105. Cor. *If a transversal is perpendicular to one of two parallel lines, it is perpendicular to the other also.*

Ex. **1.** Two $\parallel$ lines AB and CD are cut by a transversal.

(*a*) If $\angle c = 70°$, find $\angle f$.
(*b*) If $\angle c = 80°$, find $\angle e$.
(*c*) If $\angle c = 75°$, find $\angle g$.



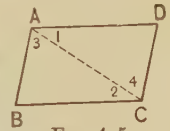
Exs. 1–3

Ex. 2. If $AB \parallel CD$, and
(*a*) $\angle a = 110°$, find $\angle f$.
(*b*) $\angle b = 80°$, find $\angle g$.
(*c*) $\angle h = 100°$, find $\angle b$.

Ex. 3. If $AB \parallel CD$, prove that
(*a*) $\angle a = \angle e$.
(*b*) $\angle b = \angle f$.
(*c*) $\angle a = \angle h$.
(*d*) $\angle c$ is the sup. of $\angle e$.
(*e*) $\angle g$ is the sup. of $\angle d$.

Ex. 4. If $AD = BC$, and $AD \parallel BC$, then $\angle 3 = \angle 4$.

Ex. 5. If $AB \parallel CD$, and $AD \parallel BC$, then $\angle B = \angle D$.

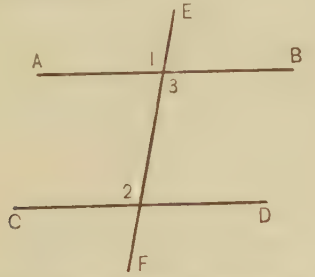


Exs. 4, 5

Proposition XIX. Theorem

106. *If two parallel lines are cut by a transversal, the corresponding angles are equal.*

[Converse of Prop. XIV.]



Given parallel lines AB and CD and the corr. $\measuredangle$ 1 and 2.

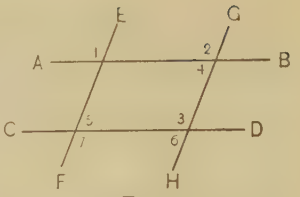
To prove $\angle 1 = \angle 2$.

Proof

STATEMENTS	REASONS
1. $\angle 1 = \angle 3$.	**1.** Vertical $\measuredangle$ are equal.
2. $\angle 2 = \angle 3$.	**2.** Alt. int. $\measuredangle$ of $\parallel$ lines are equal.
3. $\therefore \angle 1 = \angle 2$.	**3.** Things equal to the same thing are equal to each other.

Ex. 1. If $AB \parallel CD$ and $EF \parallel GH$, prove that

(a) $\angle 1 = \angle 3$.
(b) $\angle 4 = \angle 5$.
(c) $\angle 2 = \angle 7$.



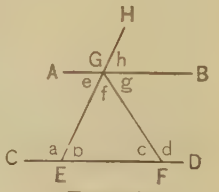
Ex. 1

Ex. 2. If $AB \parallel CD$, and EH is a st. line, find the numerical value of

(a) $\angle d$, if $\angle g = 80°$.
(b) $\angle a$, if $\angle h = 60°$.
(c) $\angle FGH$, if $\angle b = 60°$, and $\angle c = 70°$.
(d) $\angle b$, if $\angle FGH = 140°$, and $\angle c = 80°$.
(e) $\angle f$, if $\angle b = 65°$, and $\angle c = 70°$.
(f) $\angle b$, if $\angle f = 50°$, and $\angle c = 70°$.
(g) $\angle HGF$, if $\angle e = 55°$, and $\angle c = 75°$.

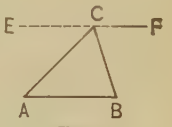
Ex. 3. If $AB \parallel CD$, prove that

(a) $\angle FGH = \angle b + \angle c$.
(b) $\angle f = 180° - \angle h - \angle c$.
(c) $\angle CEH = \angle f + \angle c$.
(d) $\angle b + \angle f + \angle c = 180°$.



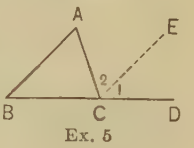
Exs. 2, 3

Ex. 4. Prove that the sum of the three angles of $\triangle ABC$ equals 180°.



Ex. 4

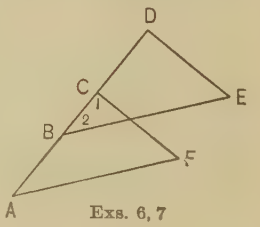
Ex. 5. If ACD is an ext. angle of $\triangle ABC$, then $\angle ACD = \angle A + \angle B$.



Ex. 5

Ex. 6. If AD is a st. line, $AB = CD$, $AF \parallel BE$, and $CF \parallel DE$, then $DE = CF$.

Ex. 7. If $AB = CD$, $CF = DE$, $CF \parallel DE$, and AD is a st. line, then $AF = BE$.

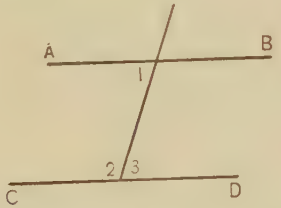


Exs. 6, 7

PROPOSITION XX. THEOREM

107. *If two parallel lines are cut by a transversal, the interior angles on the same side of the transversal are supplementary.*

[Converse of Prop. XV.]



Given $AB \parallel CD$, and the int. $\measuredangle$ 1 and 2, lying on the same side of a transversal.

To prove $\angle 1$ is the sup. of $\angle 2$.

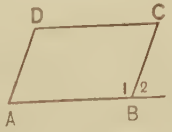
Proof

STATEMENTS	REASONS
1. $AB \parallel CD$.	**1.** By hyp.
2. $\angle 1 = \angle 3$.	**2.** Alt. int. $\measuredangle$ of $\parallel$s are $=$.
3. $\angle 3$ is sup. of $\angle 2$.	**3.** If two adj. $\measuredangle$ have their exterior sides in a st. line, they are sup.
4. $\therefore \angle 1$ is sup. of $\angle 2$.	**4.** Sub.

Ex. 1. If $AB \parallel CD$, and $AD \parallel BC$, prove

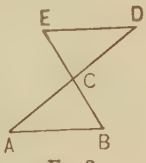
(*a*) $\angle 2$ is the sup. of $\angle D$.

(*b*) $\angle A = \angle C$.



Ex. 1

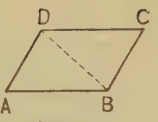
Ex. 2. If $ED \parallel AB$, $AC = CD$, and EB and AD are st. lines, $\triangle ABC \cong \triangle CDE$.



Ex. 2

Ex. 3. If the opposite sides of a quadrilateral are parallel, they are also equal.

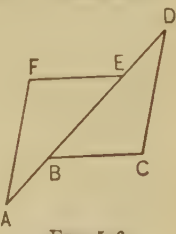
Ex. 4. If two sides of a quadrilateral are equal and parallel, the other two sides are equal and parallel.



Exs. 3, 4

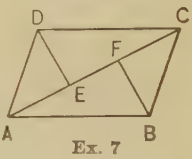
Ex. 5. If $AB = ED$, $BC \parallel FE$, $AF \parallel DC$, and AD is a st. line, $\triangle AEF$ must be congruent to $\triangle BCD$.

Ex. 6. If $AB = ED$, $BC = FE$, $BC \parallel FE$, and AD is a st. line, AF must equal CD.



Exs. 5, 6

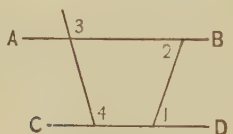
Ex. 7. If $AB \parallel CD$, $AD \parallel BC$, $AE = FC$, and AC in a st. line, then $DE \parallel BF$. (Apply Ex. 3.)



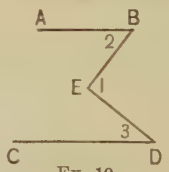
Ex. 7

Ex. 8. The bisectors of a pair of corresponding angles formed by parallel lines are parallel.

Ex. 9. In the diagram below, if $\angle 1 = \angle 2$, then $\angle 3 = \angle 4$.



Ex. 9

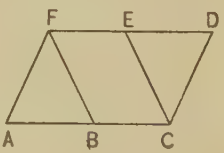


Ex. 10

* Ex. 10. If in the diagram above, $AB \parallel CD$, prove that $\angle 1 = \angle 2 + \angle 3$.

* Ex. 11. If in the diagram, $AC \parallel FD$, $AF \parallel CD$, and $FB \parallel EC$, prove that

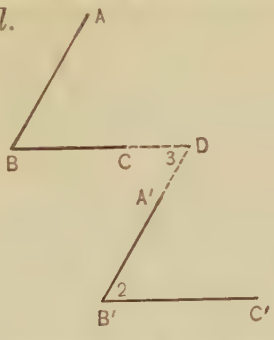
$$\triangle AFB \cong \triangle ECD.$$



Ex. 11

Proposition XXI. Theorem

108. *Two angles whose corresponding sides are parallel and extend in the same direction from their vertices are equal.*



Given $\angle B$ and $\angle B'$ with $AB \parallel A'B'$ and $BC \parallel B'C'$.

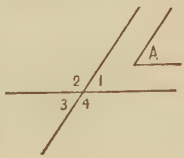
To prove $\angle B = \angle B'$.

Proof

STATEMENTS	REASONS
1. $AB \parallel A'B'$ and $BC \parallel B'C'$.	1. Hyp.
2. Produce, if necessary, BC and $B'A'$ until they meet at D.	2. Two intersecting lines cannot both be $\parallel$ to a third st. line.
3. $\angle B = \angle 3$ and $\angle B' = \angle 3$.	3. Alt. int. $\measuredangle$ of $\parallel$ lines are $=$.
4. $\therefore \angle B = \angle B'$.	4. Things $=$ to the same thing are $=$ to each other.

109. Cor. *Angles whose corresponding sides are parallel are either equal or supplementary.*

Thus, in the diagram opposite, if the corresponding sides are parallel, $\measuredangle 1$ and 3 equal $\angle A$, and $\measuredangle 2$ and 4 are sup. of $\angle A$.

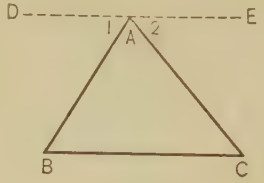


Note. Two $\measuredangle$ are equal if their sides are parallel right side to right side and left side to left side.

TRIANGLES — PART II

PROPOSITION XXII. THEOREM

110. *The sum of the angles of a triangle is equal to a straight angle.*



Given $\triangle ABC$.

To prove $\angle A + \angle B + \angle C =$ a st. $\angle$.

Proof

STATEMENTS	REASONS
1. Through A draw $DE \parallel BC$.	**1.** Through a given pt. to draw a line $\parallel$ to a given line.
2. $\angle 1 = \angle B$.	**2.** Alt. int. $\measuredangle$ of $\parallel$ lines are $=$.
3. $\angle 2 = \angle C$.	**3.** Alt. int. $\measuredangle$ of $\parallel$ lines are $=$.
4. $\angle A + \angle 1 + \angle 2 =$ a st. $\angle$.	**4.** The whole equals the sum of its parts.
5. $\therefore \angle A + \angle B + \angle C =$ a st. $\angle$.	**5.** Sub.

111. COR. 1. *In a triangle there can be at most one obtuse angle or one right angle.*

112. COR. 2. *The acute angles of a right triangle are complementary.*

113. COR. 3. *If two triangles have two angles of the one equal respectively to two angles of the other, the third angles are equal.*

114. Cor. 4. *Two triangles are congruent if two angles and the side opposite one of them are equal respectively to two angles and the corresponding side of the other* (s. a. a. = s. a. a.).

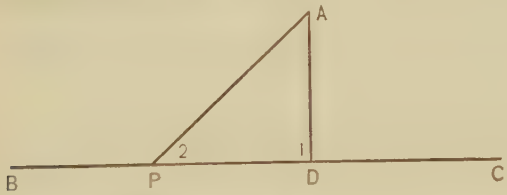
115. Cor. 5. *Each angle of an equiangular triangle is 60°.*

116. Cor. 6. *Two right triangles are congruent, if the hypotenuse and an acute angle of one triangle are equal to the hypotenuse and an acute angle of the other triangle.*

Hint. Use Cor. 4.

117. Cor. 7. *Two right triangles are congruent, if an arm and an acute angle of one triangle are equal to an arm and the corresponding acute angle of the other triangle.*

118. Cor. 8. *From a point without a line there can be only one perpendicular to that line.*



Given point A outside line BC, $AD \perp BC$, AP any other line from A to BC.

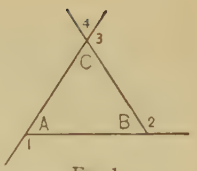
To prove AP is not $\perp$ to BC.

Proof

STATEMENTS	REASONS
1. At least one $\perp$ AD can be drawn to BC.	1. From a pt. without a line, to let fall a $\perp$ to that line (Prop. XI).
2. $\angle 1$ is a rt. $\angle$.	2. Def. of $\perp$.
3. $\angle 2$ is an acute $\angle$.	3. A right $\triangle$ has two acute $\measuredangle$.
4. $\therefore$ AP is not $\perp$ BC.	4. AP does not form rt. $\measuredangle$ with BC.

Ex. 1. Find the value of

(a) $\angle A$, if $\angle B = 60°$, and $\angle C = 50°$.
(b) $\angle B$, if $\angle A = m°$, and $\angle C = n°$.
(c) $\angle 3$, if $\angle A = 50°$, and $\angle B = 70°$.
(d) $\angle 4$, if $\angle 2 = 110°$, and $\angle A = 60°$.
(e) $\angle 1$, if $\angle 4 = 40°$, and $\angle B = 70°$.
(f) $\angle 3$, if $\angle 2 = 140°$, and $\angle 1 = 120°$.
(g) $\angle A + \angle B$, if $\angle C = m°$.



Ex. 1

Ex. 2. If C is the vertex angle of isosceles triangle ABC, find

(a) $\angle A$, if $\angle C = 40°$.
(b) $\angle B$, if $\angle C = m°$.
(c) $\angle C$, if $\angle A = 40°$.
(d) $\angle C$, if $\angle B = n°$.

Ex. 3. *How many degrees are there in each angle of an isosceles right triangle?*

Ex. 4. If two angles of a triangle are 60° and 40° respectively, what is the angle formed by the bisectors of these angles?

Ex. 5. *If two angles of a triangle are given, construct the third.*

Ex. 6. If a base angle of an isosceles triangle is given, construct the vertex angle.

Ex. 7. If the vertex angle of an isosceles triangle is given, construct a base angle.

Ex. 8. *Construct angles of* 60°; 30°; 120°; 75°.

Ex. 9. The bisectors of two interior angles on the same side of a transversal to two parallel lines are perpendicular to each other.

Ex. 10. If CB and DE are drawn respectively perpendicular upon the sides of angle A, prove $\angle ACB = \angle ADE$.

Ex. 11. If CD is the altitude upon the hypotenuse of right triangle ABC, prove $\angle ACD = \angle B$.

Ex. 12. If CD and AE are altitudes of triangle ABC, prove $\angle BCD = \angle BAE$.

Ex. 13. *Find the sum of the four angles of a quadrilateral.*

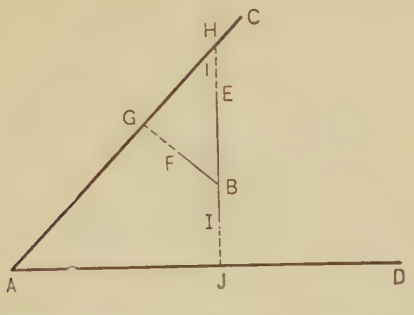
Ex. 14. If two angles of a triangle are equal, the bisector of the third angle divides the figure into two congruent triangles.

Ex. 15. The altitudes upon the arms of an isosceles triangle are equal.

Ex. 16. The lines drawn from the mid-point of the base of an isosceles triangle perpendicular to the arms are equal.

Proposition XXIII. Theorem

119. *If two angles have their sides respectively perpendicular, they are either equal or supplementary.*



Given $\measuredangle CAD$ and FBE with $AD \perp BE$, $AC \perp BF$.

To prove $\angle CAD = \angle FBE$ and $\angle CAD$ suppl. to $\angle FBI$.

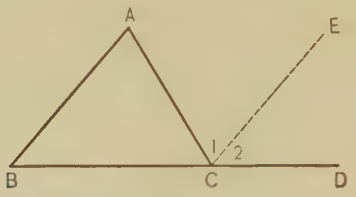
Proof

STATEMENTS	REASONS
1. Extend BF to G, BE to H, and BI to J.	**1.** A line may be extended.
2. $\measuredangle BGH$ and AJH are right $\measuredangle$.	**2.** $\perp$s form right $\measuredangle$.
3. $\angle CAD$ is comp. $\angle 1$. $\angle FBE$ is comp. $\angle 1$.	**3.** The acute $\measuredangle$ of a right $\triangle$ are complementary $\measuredangle$.
4. $\therefore \angle CAD = \angle FBE$.	**4.** Compl. of the same $\angle$ are $=$.
Also,	
5. $\angle FBE$ is sup. $\angle FBI$.	**5.** Two $\measuredangle$ are sup. if their ext. sides lie in the same st. line.
6. $\therefore \angle CAD$ sup. $\angle FBI$.	**6.** Sub.

Note. Two $\measuredangle$ whose sides are $\perp$ right to right and left to left, are equal.

Proposition XXIV. Theorem

120. *An exterior angle of a triangle is equal to the sum of its two remote interior angles.*



Given the ext. $\angle ACD$ of $\triangle ABC$.

To prove $\angle ACD = \angle A + \angle B$.

Proof

STATEMENTS	REASONS
1. Draw $CE \parallel BA$.	1. Through a pt. to draw a line $\parallel$ to a given line.
2. $\angle 1 = \angle A$.	2. Alt. int. $\measuredangle$ of $\parallel$ lines are $=$.
3. $\angle 2 = \angle B$.	3. Corr. $\measuredangle$ of $\parallel$ lines are $=$.
4. $\angle 1 + \angle 2 = \angle A + \angle B$.	4. If $=$ be added to $=$ their sums are equal.
5. $\angle ACD = \angle 1 + \angle 2$.	5. Whole $=$ sum of its parts.
6. $\therefore \angle ACD = \angle A + \angle B$.	6. Things $=$ same thing are $=$ to each other.

Ex. 1. Given two angles of a triangle, construct an angle equal to the remote exterior angle.

Ex. 2. Given an exterior angle and a remote interior angle of a triangle. Construct an angle equal to the other remote interior angle.

Ex. 3. If two angles of a triangle are equal, the bisector of the remote exterior angle is parallel to the opposite side of the triangle.

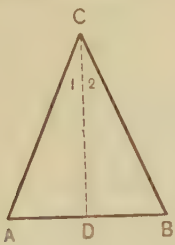
Ex. 4. If at each vertex of a triangle, one exterior angle is drawn, the sum of the three angles is equal to four right angles.

Ex. 5. The sum of two exterior angles of a triangle diminished by the third interior angle is equal to two right angles.

* Ex. 6. If the sum of two exterior angles of a triangle is equal to three right angles, the triangle is a right one.

Proposition XXV. Theorem

121. *If two angles of a triangle are equal, the sides opposite these angles are equal.* [Converse of Prop. IV.]



Given $\triangle ABC$, with $\angle A = \angle B$.

To prove $AC = BC$.

Proof

STATEMENTS	REASONS
1. Draw the bisector of $\angle C$ and extend it to meet AB at D.	1. Every $\angle$ has a bisector.
2. $\angle A = \angle B$.	2. Hyp.
3. $\angle 1 = \angle 2$.	3. Cons.
4. $CD = CD$.	4. Iden.
5. $\triangle ACD \cong \triangle BCD$.	5. s. a. a. = s. a. a.
6. $\therefore AC = BC$.	6. Corr. sides of $\cong$ $\triangle$ are =.

122. Cor. *An equiangular triangle is also equilateral.*

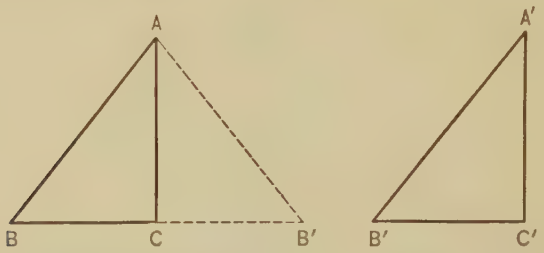
123. Method VII. **Propositions IV and XXV are occasionally used to prove the equality of lines or the equality of angles.** (Compare note on p. 31.)

Ex. 1. *If the vertex angle of an isosceles triangle is 60°, the triangle is equilateral.*

Ex. 2. The bisectors of the base angles of an isosceles triangle, together with the base, form an isosceles triangle.

Proposition XXVI. Theorem

124. *Two right triangles are congruent if the hypotenuse and an arm of the one are respectively equal to the hypotenuse and an arm of the other.* (*hy. arm = hy. arm.*)



Given $\triangle ABC$ and $A'B'C'$, with $AB = A'B'$, $AC = A'C'$, and the rt. $\angle C$ and C'.

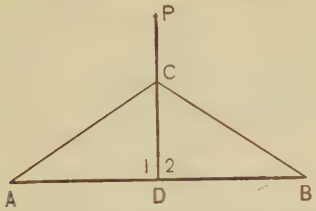
To prove $\triangle ABC \cong \triangle A'B'C'$.

Proof

STATEMENTS	REASONS
1. Place $\triangle A'B'C'$ so that $A'C'$ coincides with AC, and that B and B' fall on opposite sides of AC.	**1.** Axiom 19. Also, $AC = A'C'$, by hyp.
2. $\angle BCA$ and $B'CA$ are rt. $\angle$.	**2.** Hyp.
3. $\angle BCB'$ is a st. $\angle$.	**3.** A st. $\angle$ = sum of 2 right $\angle$.
4. BC and CB' form a st. line BCB'.	**4.** Def. st. $\angle$.
5. $BCB'A$ is a $\triangle$, with $AB = AB'$.	**5.** $AB = A'B'$, by hyp.
6. $\angle B = \angle B'$.	**6.** The base $\angle$ of an isos. $\triangle$ are $=$.
7. $\therefore \triangle ABC \cong \triangle A'B'C'$.	**7.** (116).

Proposition XXVII. Theorem

125. *Every point in the perpendicular-bisector of a line is equidistant from the extremities of that line.*



Given PD, the perpendicular-bisector of AB, and C any point in PD.

To prove $CA = CB$.

Proof

STATEMENTS	REASONS
1. $AD = DB$.	1. By hyp., PD bisects AB.
2. $PD \perp AB$.	2. Hyp.
3. $\angle 1 = \angle 2$.	3. $\left\{\begin{array}{l}\text{Lines form rt. } \measuredangle.\\ \text{Right } \measuredangle \text{ are } =.\end{array}\right.$
4. $DC = DC$.	4. Iden.
5. $\triangle ADC \cong \triangle BDC$.	5. s. a. s. = s. a. s.
6. $\therefore CA = CB$.	6. Corr. sides of $\cong$ $\triangle$ are =.

Note. The converse theorem is the corollary on p. 36.

Ex. 1. In a given line AB find a point equidistant from two given points P and Q.

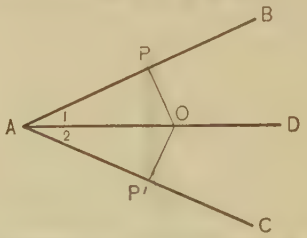
Ex. 2. In a given circle find a point equidistant from two given points.

Ex. 3. *Find a point equidistant from three given points.*

Ex. 4. Given three fixed points A, B, and C. Required to construct a point X, so that $AX = BX$, and $CX = \frac{1}{2}$ inch.

Proposition XXVIII. Theorem

126. *Every point in the bisector of an angle is equidistant from the sides of the angle.*



Given AD the bisector of $\angle BAC$, and O any point in AD and $OP \perp AB$, $OP' \perp AC$.

To prove O is equidistant from AB and AC (*i.e.* $OP = OP'$).

Proof

STATEMENTS	REASONS
1. $\angle 1 = \angle 2$.	**1.** By hyp., AD bis. $\angle BAC$.
2. $OP \perp AB$, $OP' \perp AC$.	**2.** Hyp.
3. $\triangle APO$ and $AP'O$ are rt. $\triangle$.	**3.** $\perp$s form rt. $\triangle$.
4. $AO = AO$.	**4.** Iden.
5. $\triangle APO \cong \triangle AP'O$.	**5.** Hy. $\angle$ acute = hy. $\angle$ acute.
6. $\therefore OP = OP'$.	**6.** Corr. sides of $\cong$ $\triangle$ are $=$.

127. Cor. *Every point equidistant from the sides of an angle is in the bisector of that angle.*

[Converse to Prop. XXVIII.]

Hint. Student should write out carefully the proof of this corollary. Prove $\triangle APO$ and $AP'O$ are congruent and consequently $\angle 1 = \angle 2$.

Ex. 1. In a given line AB to find a point equidistant from the sides of the given $\angle DEF$.

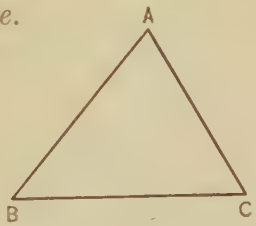
Ex. 2. In a given circle find a point equidistant from two given lines.

Ex. 3. *Find a point equidistant from three given lines.*

UNEQUAL LINES AND UNEQUAL ANGLES

Proposition XXIX. Theorem

128. *The sum of two sides of a triangle is greater than the third side.*



Given $\triangle ACB$.

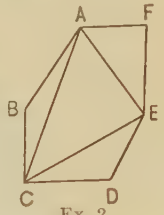
To prove $AB + AC > BC$.

Hint. See Axiom 14.

Ex. 1. Is it possible to draw a triangle whose sides are

(*a*) 4 in., 5 in., 10 in.?
(*b*) 8 in., 19 in., 12 in.?
(*c*) 4 ft., 12 ft., 8 ft.?
(*d*) 6 in., 2 in., 5 in.?

Ex. 2. Prove that the perimeter of polygon $ABCDEF >$ perimeter of $\triangle ACE$.



Ex. 2

Ex. 3. If in $\triangle ABC$, $AC = BC$, and a point D in the prolongation of AC be connected with B, then $DA > DB$.

Ex. 4. If in the side BC of $\angle ABC$ any point D be taken, then $AB + BC > AD + DC$.

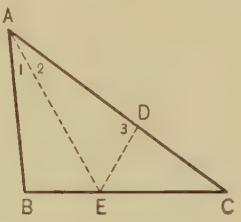
Ex. 5. If from any point E in $\triangle ABC$, EB and EC are drawn, then $AB + AC > EB + EC$.

Hint. Produce BE until it meets AC in D.

Ex. 6. *The difference of any two sides of a triangle is less than the third side.*

Proposition XXX. Theorem

129. *If two sides of a triangle are unequal, the angles opposite are unequal, and the greater angle is opposite the greater side.*



Given $\triangle ABC$ with $AC > AB$.

To prove $\angle B > \angle C$.

Proof

STATEMENTS	REASONS
1. Draw the angle-bisector AE.	1. Every $\angle$ has a bisector.
2. On AC lay off $AD = AB$.	2. A line may be laid off on another line.
3. Draw DE.	3. Ax. 13.
4. $AB = AD$.	4. Con.
5. $AE = AE$.	5. Iden.
6. $\angle 1 = \angle 2$.	6. Con.
7. $\triangle ABE \cong \triangle ADE$.	7. s. a. s. = s. a. s.
8. $\angle B = \angle 3$.	8. Corr. $\measuredangle$ of congruent $\triangle$ are =.
9. $\angle 3 > \angle C$.	9. (88).
10. $\therefore \angle B > \angle C$.	10. Sub.

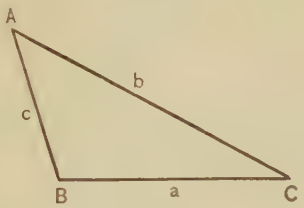
Ex. 1. If one arm AC of an isosceles triangle ABC is produced to D, then $\angle ABD > \angle ADB$.

Ex. 2. In $\triangle ABC$, $AB > AC$, and $\angle A = 60°$. Which is the greatest angle of the triangle?

PROPOSITION XXXI. THEOREM

130. *If two angles of a triangle are unequal, the sides opposite are unequal, and the greater side is opposite the greater angle.*

[Converse of Prop. XXX.]



Given $\triangle ABC$, with $\angle B > \angle C$.

To prove $b > c$.

Proof

STATEMENTS	REASONS
1. Either $b < c$, or $b = c$, or $b > c$.	1. Only 3 suppositions are possible.
2. If $b < c$, then $\angle B < \angle C$.	2. In a $\triangle$, the greater $\angle$ lies opp. the greater side.
3. This is impossible.	3. By hyp., $\angle B > \angle C$.
4. If $b = c$, then $\angle B = \angle C$.	4. The base $\measuredangle$ of an isos. $\triangle$ are equal.
5. This is impossible.	5. By hyp., $\angle B > \angle C$.
6. $\therefore b > c$.	6. The other two suppositions are false.

131. COR. *The perpendicular is the shortest line that can be drawn from a point to a given line.* (118)

NOTE. The method used in the above proof is known as the **indirect method.** Instead of showing that a certain conclusion is true, we examine *all* conclusions which contradict the one to be proved, and demonstrate that these are false. Thus, in order to demonstrate $a = b$, we simply prove that the statement $a \neq b$ is false. Or to prove $m > n$, we disprove the only contradictory conclusions, viz. $m = n$, and $m < n$.

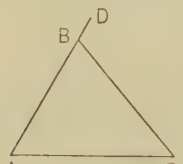
Ex. 1. *Which is the greatest side of a right triangle? Of an obtuse triangle?*

Ex. 2. If two angles of a triangle are 50° and 60° respectively, which is the greater of the two opposite sides? Which is the greatest side of the triangle? Which is the shortest?

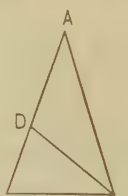
Ex. 3. If in $\triangle ABC$, $AB > AC$ and $\angle B = 60°$, which is the greatest side of the triangle? Which is the smallest?

Ex. 4. If in $\triangle ABC$, $AB > AC$, and $\angle A = 60°$, which is the greatest side of the triangle?

Ex. 5. If $\angle A$ of $\triangle ABC$ equals two thirds of a right angle, and the exterior angle DBC equals 110°, which is the greatest side of the triangle? Which is the shortest?



Ex. 5

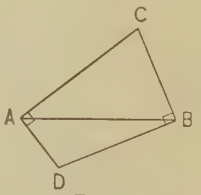


Ex. 6

Ex. 6. If in $\triangle ABC$, $AB = AC$, prove that $DC > DB$.

Ex. 7. In $\triangle ABC$, $AC > BC$. If the bisectors of angles A and B meet in D, prove $AD > BD$.

Ex. 8. $AC > BC$, $AD \perp AC$, and $DB \perp BC$. Prove $BD > AD$.



Ex. 8

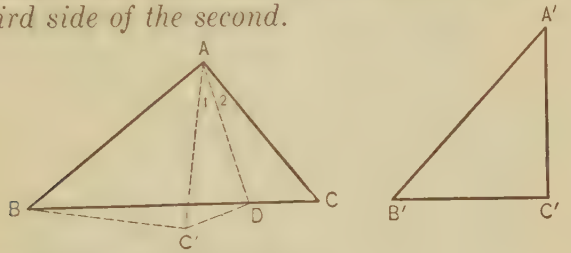
Ex. 9. Prove Prop. XXXI directly.

Hint. At B draw $\angle CBD = \angle C$.

Ex. 10. The sum of the altitudes of a triangle is less than the perimeter of the triangle.

Proposition XXXII. Theorem

132. *If two triangles have two sides of the one equal respectively to two sides of the other, but the included angle of the first greater than the included angle of the second, then the third side of the first is greater than the third side of the second.*



Given $\triangle ABC$ and $A'B'C'$ with

$$AB = A'B',\ AC = A'C',\ \angle A > \angle A'.$$

To prove $BC > B'C'$.

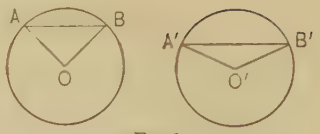
Proof

STATEMENTS	REASONS
1. Apply $\triangle A'B'C'$ to $\triangle ABC$ so that $A'B'$ coincides with AB.	1. By hyp., $AB = A'B'$ and Ax. 19.
2. $A'C'$ will fall between AB and AC as AC'.	2. By hyp., $\angle A > A'$.
3. Draw AD bisecting $\angle CAC'$ to meet BC in D, and draw DC'.	3. Every $\angle$ has a bisector. To draw a st. line.
4. $AC' = AC$.	4. Hyp.
5. $\angle 1 = \angle 2$.	5. Con.
6. $AD = AD$.	6. Iden.
7. $\triangle C'AD \cong \triangle CAD$.	7. s. a. s. = s. a. s.
8. $DC' = DC$.	8. Why?
9. $BD + DC' > BC'$.	9. (128).
10. $BD + DC > BC'$.	10. Sub.
11. $\therefore BC > BC'$ or $B'C'$.	11. Sub.

Ex. **1**. If O and O' are two equal circles, and

$$\angle A'O'B' > \angle AOB,$$

then $$A'B' > AB.$$



Ex. 1

Ex. **2**. If in triangle ACB the median AD makes $\angle ADB$ an acute $\angle$, prove

(*a*) $AC > AB$.

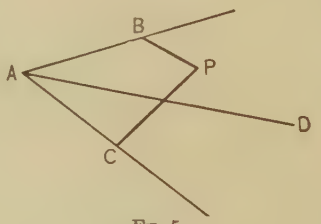
(*b*) $\angle B > \angle C$.

Ex. 2

Ex. **3**. If in $\triangle ABC$, $\angle A = \angle B$, and a point D in AB be taken so that $\angle ACD > \angle DCB$, then $AD > DB$.

Ex. **4**. *Every point not in the perpendicular bisector of a line is unequally distant from the ends of the line.*

Hint. Apply Prop. XXXII.

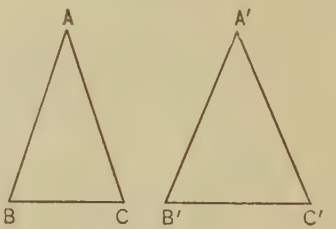


Ex. 5

Ex. **5**. AD is the bisector of $\angle BAC$ and $AB = AC$. Prove

$$PC > PB.$$

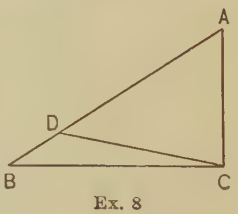
Ex. **6**. $\triangle ABC$ and $\triangle A'B'C'$ are isosceles $\triangle$ with bases BC and $B'C'$. If $AB = A'C'$ and $\angle B > \angle B'$, which $\triangle$ has the greater base?



Ex. 6

Ex. **7**. Prove Prop. XXXII when point C' falls inside $\triangle ABC$.

Ex. **8**. In $\triangle BCA$, $AC \perp BC$ and $AD = BC$. Prove $AB > DC$.

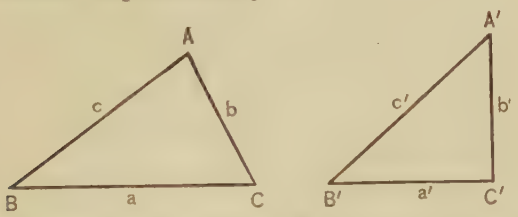


Ex. 8

Proposition XXXIII. Theorem

133. *If two triangles have two sides of the one equal respectively to two sides of the other, but the third side of the first greater than the third side of the second, then the included angle of the first is greater than the included angle of the second.*

[Converse of Prop. XXXII.]



Given $\triangle ABC$ and $A'B'C'$, with $b = b'$, $c = c'$, and $a > a'$.

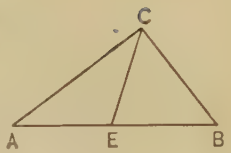
To prove $\angle A > \angle A'$.

Proof

STATEMENTS	REASONS
1. Either $\angle A < \angle A'$, or $\angle A = \angle A'$, or $\angle A > \angle A'$.	1. Only 3 suppositions are possible.
2. If $\angle A < \angle A'$, then $a < a'$.	2. If two $\triangle$ have 2 sides of the one equal respectively to 2 sides of the other, but the included $\angle$ of the first $>$ the included $\angle$ of the second, then the 3d side of the first $>$ 3d side of the second.
3. This is impossible.	3. $a > a'$, by hyp.
4. If $\angle A = \angle A'$, then $\triangle ABC \cong \triangle A'B'C'$.	4. s. a. s. = s. a. s.
5. $a = a'$.	5. Corr. sides, etc.
6. This is impossible.	6. $a > a'$, by hyp.
7. $\therefore \angle A > \angle A'$.	7. The other two suppositions have been proved false.

Ex. 1. If in $\triangle ABC$, having $AC > BC$, the median CE is drawn, then $\angle AEC$ is an obtuse angle.

Ex. 2. Given $\triangle ABC$, having $\angle A = \angle B$. If the point D in AB be taken so that $AD > DB$, then $\angle ACD > \angle DCB$.

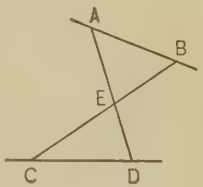


Ex. 1

134. METHOD VIII. **Inequality of lines is generally proved by means of Props. XXIX, XXXI, XXXII. If it is impossible to discover any relation of the angles, Prop. XXIX is used; if the two lines are parts of the same triangle, and the opposite angles can be proved unequal, Prop. XXXI is used; but if the sides or angles are parts of different triangles, Prop. XXXII is used. In some cases several methods will lead to the desired result.**

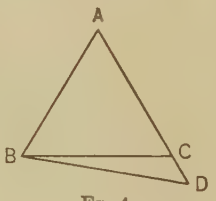
The inequality of angles is proved in a similar manner by Props. XII, XXX, and XXXIII.

Ex. 3. If AD and BC intersect in E, prove $AD + BC > AB + CD$. (Which of the three propositions must be used?)



Ex. 3

Ex. 4. If AC, a side of an equilateral triangle, is produced to D, and BD is joined, then $AD > BD > AB$.

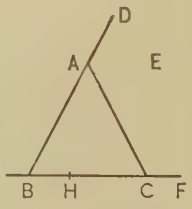


Ex. 4

Ex. 5. If $AB = AC$, prove that

(*a*) $BD > DC$. (*c*) $AF > AB$.

(*b*) $BE > EC$. (*d*) $AB > AH$.



Ex. 5

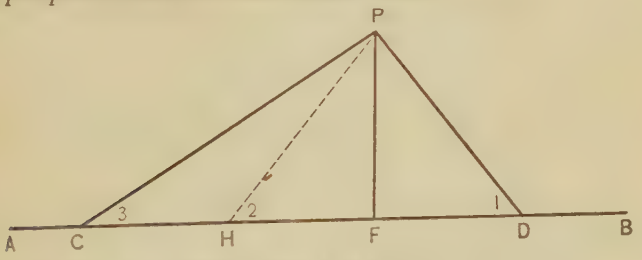
Ex. 6. The sum of the diagonals of any quadrilateral is less than the perimeter, but greater than the semiperimeter of the quadrilateral.

* Ex. 7. If in $\triangle ABC$, having $AB < BC$, BD, the bisector of angle B, is drawn, meeting AC in D, then $AD < DC$.

* Ex. 8. If in $\triangle ABC$, having $AC > BC$, the median CD is drawn, then any point E in CD is nearer to B than to A.

Proposition XXXIV. Theorem

135. *If, from a point in a perpendicular to a given line, two oblique lines are drawn to the line, the one cutting off the greater distance from the foot of the perpendicular is the longer line.*



Given line AB, point P, $PF \perp AB$, $CF > FD$.

To prove $PC > PD$.

Proof

STATEMENTS	REASONS
1. On FC lay off $FH = FD$.	1. To lay off a given length on a line.
2. Draw HP.	2. A st. line can be drawn between two pts.
3. $PH = PD$.	3. Every pt. in the $\perp$ bisector of a line is equidistant from the ends of the line.
4. $\angle 2 = \angle 1$.	4. If two sides of a $\triangle$ are $=$, the $\measuredangle$ opp. are $=$.
5. $\angle 2 > \angle 3$.	5. An ext. $\angle$ of a $\triangle >$ either remote int. $\angle$.
6. $\angle 1 > \angle 3$.	6. Sub.
7. $\therefore PC > PD$.	7. If two $\measuredangle$ of a $\triangle$ are unequal, opp. the greater side lies the greater $\angle$.

Ex. 1. *State and prove the converse of Prop. XXXIV.*

Ex. 2. *Prove that from a given point outside a given line only two equal oblique lines can be drawn to the given line.*

QUADRILATERALS

136. DEF. A **quadrilateral** is a polygon of four sides.

A **trapezoid** is a quadrilateral having two, and only two, sides parallel.

A **parallelogram** is a quadrilateral having its opposite sides parallel.

137. DEF. A **rectangle** is a parallelogram whose angles are right angles.

A **square** is an equilateral rectangle.

A **rhombus** is an equilateral parallelogram.

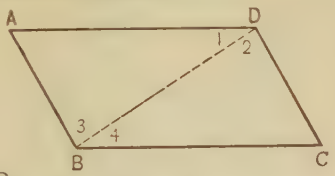
138. DEF. An **isosceles trapezoid** is a trapezoid whose nonparallel sides are equal. The parallel sides of a trapezoid are called its **bases**, and are distinguished as **upper** and **lower**.

The **median** of a trapezoid is the line joining the mid-points of the nonparallel sides.

A **diagonal** of a quadrilateral is a straight line joining opposite vertices. The **altitude** of a parallelogram or trapezoid is the perpendicular distance between the two bases.

Proposition XXXV. Theorem

139. *The opposite sides and the opposite angles of a parallelogram are equal.*



Given $\square ABCD$.

To prove $AD = BC$, $AB = CD$, $\angle A = \angle C$, $\angle B = \angle D$.

Proof

STATEMENTS	REASONS
1. Draw BD.	1. A st. line can be drawn between two pts.
2. $AD \parallel BC$, $AB \parallel DC$.	2. The opp. sides of a $\square$ are $\parallel$.
3. $\angle 1 = \angle 4$, $\angle 3 = \angle 2$.	3. Alt. int. $\angle$s of $\parallel$ lines are $=$.
4. $BD = BD$.	4. Iden.
5. $\triangle ABD \cong \triangle BDC$.	5. a. s. a. $=$ a. s. a.
6. $AD = BC$. $AB = CD$. $\angle A = \angle C$.	6. Corr. parts of $\cong$ $\triangle$ are $=$.
7. In like manner by drawing AC, we can prove $\angle B = \angle D$.	7. Steps similar to 1–6.

140. Cor. 1. *A diagonal divides a parallelogram into two congruent triangles.*

141. Cor. 2. *Parallels included between parallels are equal.*

142. Cor. 3. *Parallel lines are everywhere equidistant.*

Ex. 1. *The diagonals of a rectangle are equal.*

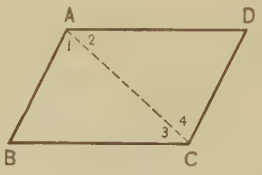
Ex. 2. *If the diagonals of a parallelogram are equal, the figure is a rectangle.*

Ex. 3. The perpendiculars to a diagonal of a parallelogram from the opposite vertices are equal.

Proposition XXXVI. Theorem

143. *If the opposite sides of a quadrilateral are equal, the figure is a parallelogram.*

[Converse of Prop. 35.]



Given quadrilateral $ABCD$ with

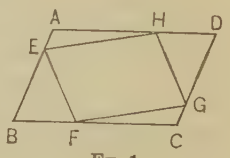
$$AB = CD,\ AD = BC.$$

To prove $ABCD$ is a $\square$.

Proof

STATEMENTS	REASONS
1. Draw AC.	**1.** A st. line can be drawn between two points.
2. $AB = CD,\ AD = BC$.	**2.** Hyp.
3. $AC = AC$.	**3.** Iden.
4. $\triangle ABC \cong \triangle ACD$.	**4.** s. s. s. = s. s. s.
5. $\angle 1 = \angle 4,\ \angle 3 = \angle 2$.	**5.** Corr. $\measuredangle$ of congruent $\triangle$ are =.
6. $AB \parallel CD,\ BC \parallel AD$.	**6.** If a pair of alt. int. $\measuredangle$ are =, the lines are $\parallel$.
7. $\therefore$ $ABCD$ is a $\square$.	**7.** Its opp. sides are parallel.

Ex. 1. If $ABCD$ is a parallelogram, $AE = CG$, and $BF = DH$, then $EFGH$ is a parallelogram.

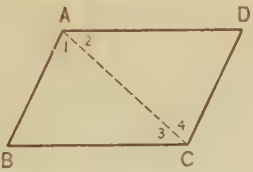


Ex. 1

Ex. 2. The lines joining the mid-points of the sides of a parallelogram, taken in order, form a parallelogram.

Proposition XXXVII. Theorem

144. *If two sides of a quadrilateral are equal and parallel, the figure is a parallelogram.*



Given quadrilateral $ABCD$ with

$$AD \parallel BC,\ AD = BC.$$

To prove $ABCD$ is a $\square$.

Proof

STATEMENTS	REASONS
1. Draw AC.	1. Ax. 13.
2. $AD \parallel BC$.	2. Hyp.
3. $\angle 2 = \angle 3$.	3. Alt. int. $\measuredangle$ of $\parallel$ lines are $=$.
4. $AD = BC$.	4. Hyp.
5. $AC = AC$.	5. Iden.
6. $\triangle ADC \cong \triangle ABC$.	6. s. a. s. $=$ s. a. s.
7. $\angle 4 = \angle 1$.	7. Corr. $\measuredangle$ of $\cong$ $\triangle$ are $=$.
8. $BA \parallel CD$.	8. (93).
9. $\therefore$ $ABCD$ is a $\square$.	9. Both pairs of opp. sides are $\parallel$.

Ex. 1. If on opposite sides of AB the parallelograms $ABCD$ and $ABEF$ are drawn, then $CDFE$ is also a parallelogram.

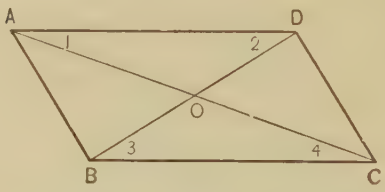
Ex. 2. If two opposite sides of a parallelogram are produced by the same length in opposite directions and their ends joined to the nearest vertices, another parallelogram is formed.

145. Method IX. **Lines may be shown to be parallel by proving them to be opposite sides of a parallelogram.**

Ex. 3. If two opposite sides of a parallelogram are divided into three equal parts, and the respective points of division are joined, the lines are parallel

Proposition XXXVIII. Theorem

146. *The diagonals of a parallelogram bisect each other.*



Given $\square ABCD$ having diagonals AC and BD.

To prove $AO = OC,\ BO = OD$.

Proof

STATEMENTS	REASONS
1. $AD \parallel BC,\ AB \parallel DC$.	1. By hyp., the fig. is a $\square$.
2. $\angle 1 = \angle 4,\ \angle 2 = \angle 3$.	2. Alt. int. $\measuredangle$ of $\parallel$ lines are $=$.
3. $AD = BC$.	3. Opp. sides of $\square$ are $=$.
4. $\triangle AOD \cong \triangle BOC$.	4. a. s. a. $=$ a. s. a.
5. $\therefore AO = OC,\ BO = OD$.	5. Corr. sides of $\cong$ $\triangle$ are $=$.

Ex. 1. *If the diagonals of a quadrilateral bisect each other, the figure is a parallelogram.* [Converse to Prop. 38.]

Ex. 2. *The diagonals of a rhombus are perpendicular to each other.*

Ex. 3. *If the diagonals of a parallelogram are perpendicular to each other, the figure is a rhombus.*

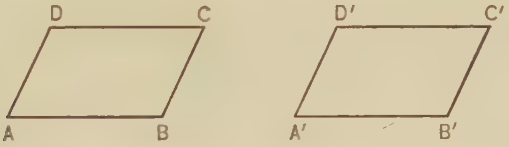
Ex. 4. If each half of the diagonals of a parallelogram is bisected and the mid-points are joined in order, another parallelogram is formed.

Ex. 5. If a diagonal bisects an angle of a parallelogram, the figure is a rhombus.

Ex. 6. If the ends of two diameters of a circle be joined in succession, a rectangle is formed.

Proposition XXXIX. Theorem

147. *Two parallelograms are congruent if two sides and the included angle of one are equal, respectively, to two sides and the included angle of the other.*



Given $\square ABCD$ and $A'B'C'D'$ with $AB = A'B'$, $AD = A'D'$, $\angle A = \angle A'$.

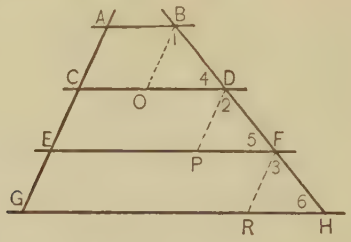
To prove $\square ABCD \cong \square A'B'C'D'$.

Proof

STATEMENTS	REASONS
1. Place $\square ABCD$ upon $A'B'C'D'$, so that AB coincides with $A'B'$.	1. Axiom 19. By hyp., $AB = A'B'$.
2. Then AD falls along $A'D'$.	2. By hyp., $\angle A = \angle A'$.
3. D coincides with D'.	3. By hyp., $AD = A'D'$.
4. BC falls along $B'C'$.	4. Axiom 16.
5. C must lie in $B'C'$ or in $B'C'$ produced.	5. Step 4.
6. Similarly, C must lie in $D'C'$ or in $D'C'$ produced.	6. Axiom 16.
7. Hence C coincides with C', and $\square ABCD$ coincides with $\square A'B'C'D'$.	7. Axiom 13 *a*.
8. $\therefore \square ABCD \cong \square A'B'C'D'$.	8. Figures are $\cong$ if they can be made to coincide.

Proposition XL. Theorem

148. *If three or more parallels intercept equal segments on one transversal, they intercept equal segments on every transversal.*



Given $AB \parallel CD \parallel EF \parallel GH$, and $AC = CE = EG$.

To prove $BD = DF = FH$.

Proof

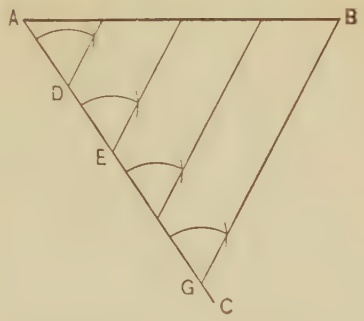
STATEMENTS	REASONS
1. Draw BO, DP, and $FR \parallel AG$.	1. (100).
2. $BO \parallel DP \parallel FR$.	2. Lines $\parallel$ to the same line are $\parallel$ to each other.
3. $\begin{cases} \angle 1 = \angle 2 = \angle 3. \\ \angle 4 = \angle 5 = \angle 6. \end{cases}$	3. Corr. $\measuredangle$ of $\parallel$ lines are $=$.
4. $BO = AC$, $DP = CE$, $FR = EG$.	4. (141).
5. $AC = CE = EG$.	5. Hyp.
6. $BO = DP = FR$.	6. Ax. 1.
7. $\triangle BOD \cong \triangle DPF \cong \triangle FRH$.	7. s. a. a. = s. a. a.
8. $\therefore BD = DF = FH$.	8. Corr. sides of $\cong$ $\triangle$ are $=$.

149. Cor. 1. *The line parallel to one side of a triangle, bisecting another side, bisects the third side.*

150. Cor. 2. *The line that bisects a nonparallel side of a trapezoid and is parallel to the bases bisects the other nonparallel side.*

PROPOSITION XLI. PROBLEM

151. *To divide a given line into any number of equal parts.*



Given line AB.

Required to divide AB into n equal parts.

Construction. From A draw any line AC.

On AC lay off any segment, as AD, n times in succession, as AD, DE, etc. Join the last point of division, G, with B, and through the other points of division draw ∥s to GB.

NOTE. Proof: Through A draw ∥ to GB, then apply Prop. 40.

Ex. Divide a line into three equal parts. Into five equal parts.

152. METHOD X. **To prove that a line is twice as large as another, we usually double the smaller and prove that its double equals the greater, or sometimes we bisect the greater and prove that its half equals the smaller. The same relation between angles is proved in a similar way.**

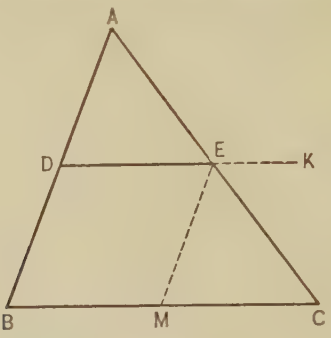
Ex. **1.** If two angles of a triangle are 30° and 60° respectively, the side opposite the angle of 30° is one half the side included by the two angles.

Ex. **2.** In triangle ABC, if through the mid-point D of AC a line DE is drawn parallel to AB, and E lies in CB, then DE is one half AB.

Ex. **3.** The angle formed by the base of an isosceles triangle and the altitude upon one arm is one half the vertical angle.

Proposition XLII. Theorem

153. *A line which joins the mid-points of two sides of a triangle is parallel to the third side and equal to half of it.*



Given $\triangle ABC$ with $AD = DB$, $AE = EC$.

To prove $DE \parallel BC$, and $DE = \frac{1}{2} BC$.

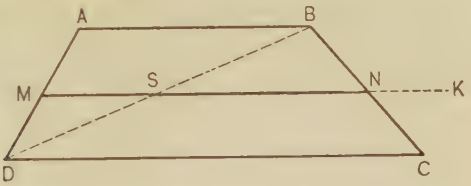
Proof

STATEMENTS	REASONS
1. Draw $DK \parallel BC$.	1. (100).
2. DK bisects AC (*i.e.* passes through point E).	2. Prop. 40, Cor. 1.
3. DE coincides with DK.	3. Ax. 13.
4. $\therefore DE \parallel BC$.	4. It is the same line as DK.
5. Draw $EM \parallel AB$.	5. Same as 1.
6. $BM = MC$.	6. Same as 2.
7. $DE = BM$.	7. Prop. 35, Cor. 2.
8. $\therefore DE = \frac{1}{2}(BC)$.	8. By 6, 7, $DE = BM$, which is one half of BC.

Ex. 1. If the mid-points of the three sides of a triangle are joined, four congruent triangles are formed.

Ex. 2. A line joining the mid-points of two adjacent sides of a quadrilateral is equal and parallel to the line joining the mid-points of the other two.

Ex. 3. *The median of a trapezoid is parallel to the bases, and equal to half the sum of the bases.*



Given trapezoid $ABCD$ with median MN.

To prove $\begin{cases} MN \parallel DC \parallel AB. \\ MN = \frac{1}{2}(AB + DC). \end{cases}$

Proof

STATEMENTS	REASONS
1. Draw $MK \parallel DC$. It is $\parallel$ to AB.	1. Why?
2. MK passes through N.	2. Prop. 40.
3. $MN \parallel DC$, also $\parallel AB$.	3. Why?
4. Draw diag. BD.	4. Why?
5. S is mid-point of DB.	5. Prop. 40, Cor. 1.
6. $MS = \frac{1}{2}AB$, $SN = \frac{1}{2}DC$.	6. Prop. 42.
7. $MN = \frac{1}{2}(AB + DC)$.	7. Why?

Ex. 4. If the mid-points of the arms of an isosceles triangle are joined to the mid-points of the base, a rhombus is formed.

154. DEF. An **equiangular polygon** is a polygon having all of its angles equal. An **equilateral polygon** is a polygon having all of its sides equal.

A polygon of five sides is called a **pentagon**; six sides, a **hexagon**; seven sides, a **heptagon**; eight sides, an **octagon**; ten sides, a **decagon**; twelve sides, a **dodecagon.**

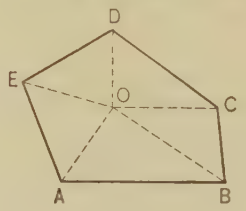
155. *Polygons that are mutually equiangular and equilateral are* ***congruent****, for they can be made to coincide.*

Ex. 5. Draw two mutually equiangular quadrilaterals that are not mutually equilateral.

Ex. 6. Draw two mutually equilateral quadrilaterals that are not mutually equiangular.

Proposition XLIII. Theorem

156. *The sum of all the angles of a polygon of n sides is equal to $(n-2)$ straight angles.*



Given $ABCDE\cdots$, a polygon of n sides.

To prove $\angle A + \angle B + \angle C + \angle D + \cdots = (n-2)$ st. $\measuredangle$.

Proof

STATEMENTS	REASONS
1. Connect any point within, as O, with all the vertices.	1. Why?
2. n $\triangle$ are formed.	2. Each side of polygon is a base.
3. Sum of $\measuredangle$ of poly. = sum all base $\measuredangle$ of all $\triangle$.	3. Ax. 10.
4. Sum of all $\measuredangle$ of $\triangle$ = n st. $\measuredangle$.	4. Why?
5. Sum of vertex $\measuredangle$ of $\triangle$ = 2 st. $\measuredangle$.	5. Why?
6. Sum of base $\measuredangle$ of $\triangle$ = $(n-2)$ st. $\measuredangle$.	6. Why?
7. $\therefore$ Sum all $\measuredangle$ of poly. = $(n-2)$ st. $\measuredangle$.	7. Why?

i.e. $\angle A + \angle B + \angle C + \cdots = (n-2)$ st. $\measuredangle$.

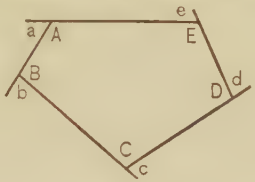
157. Cor. *Each angle of an equiangular polygon of n sides equals* $\frac{n-2}{n}$ *of a straight angle.*

Ex. 1. How many sides has a polygon the sum of whose angles is 1000 st. $\measuredangle$? 200 rt. $\measuredangle$? 24 rt. $\measuredangle$? 720°?

Ex. 2. How many degrees are in each angle of an equiangular quadrilateral? Pentagon? Hexagon? Decagon?

Proposition XLIV. Theorem

158. *If the sides of any polygon be successively produced, forming one exterior angle at each vertex, the sum of these angles is equal to two straight angles.*



Given $ABCDE$, a polygon of n sides, with the ext. $\measuredangle$ a, b, c, etc.

To prove $\angle a + \angle b + \angle c + \cdots = 2$ st. $\measuredangle$.

Proof. $\angle A + \angle a = 1$ st. $\angle$, $\angle B + \angle b = 1$ st. $\angle$, etc. (50)

$\therefore (\angle A + \angle B + \cdots) + (\angle a + \angle b + \cdots) = n$ st. $\measuredangle$. (Ax. 2.)

But $(\angle A + \angle B + \cdots) = (n-2)$ st. $\measuredangle$. (156)

$\therefore \angle a + \angle b + \angle c + \cdots = 2$ st. $\measuredangle$. (Ax. 3.)

159. Cor. *Each exterior angle of an equiangular polygon of n sides equals the $\frac{2}{n}$ part of a st. $\angle$.*

Ex. 1. How many degrees are in each exterior angle of an equiangular polygon of 10 sides? Of 9 sides? Of 36 sides? Of 72 sides?

Ex. 2. How many sides has a polygon each of whose exterior angles equals 30°? One right angle? 60°? 45°?

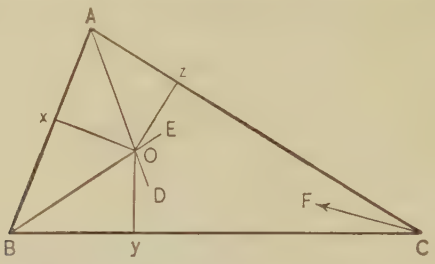
Ex. 3. How many sides has a polygon each of whose interior angles equals 160°? 179°? 135°? $\frac{1}{4}$ rt. $\measuredangle$?

Ex. 4. How many sides has an equiangular polygon, three of whose exterior angles are together equal to 90°? $\frac{3}{4}$ of a right $\angle$?

Ex. 5. How many sides has a polygon, the sum of whose interior angles equals twice the sum of its exterior angles?

PROPOSITION XLV. THEOREM

160. *The bisectors of the angles of a triangle meet in a point which is equidistant from the sides of the triangle.*



Given $\triangle ABC$, and AD, BE, and CF, the bisectors of $\measuredangle A$, B, and C respectively.

To prove (*a*) AD, BE, and CF meet in a point, as O.
(*b*) O is equidistant from AB, BC, and CA.

Proof. Since AD and BE cannot be $\parallel$, they meet in a point, as O. (107)

From O draw OX, OY, and OZ respectively $\perp AB$, BC, and CA.

$OX = OZ$. (126)

$OX = OY$. (126)

$\therefore OZ = OY$. (Ax. 1.)

$\therefore O$ lies in CF or its prolongation. (127)

$\therefore AD$, BE, and CF meet in point O, and O is equidistant from AB, BC, and CA.

161. DEF. Three or more lines are **concurrent** if they meet in a common point.

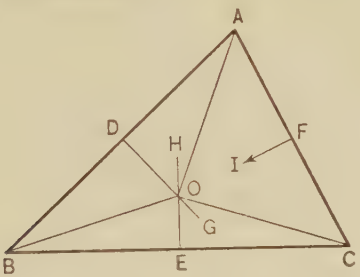
Ex. 1. *Find by a construction a point equidistant from the three sides of a triangle.*

Ex. 2. Prove that the bisectors of two exterior angles and the bisector of the third interior angle of a triangle meet in a point.

Ex. 3. Construct a point equidistant from the sides AB, BC, and CD of parallelogram $ABCD$.

Proposition XLVI. Theorem

162. *The perpendicular bisectors of the sides of a triangle meet in a point which is equidistant from the vertices of the triangle.*



Given $\triangle ABC$, and DG, EH, and FI the perpendicular-bisectors of AB, BC, and CA respectively.

To prove (*a*) DG, EH, and FI meet in a common point. (*b*) The point of intersection is equidistant from A, B, and C.

Proof. DG and EH intersect at a point, as O. (102)

$OA = OB$ and $OB = OC$. (125)

$\therefore OA = OC$. (Ax. 1.)

$\therefore O$ lies in the $\perp$ bisector of AC. (80)

Or DG, EH, and FI meet in O; and O is equidistant from A, B, and C since $OA = OB = OC$.

Ex. 1. Construct a point O equidistant from three vertices of a quadrilateral.

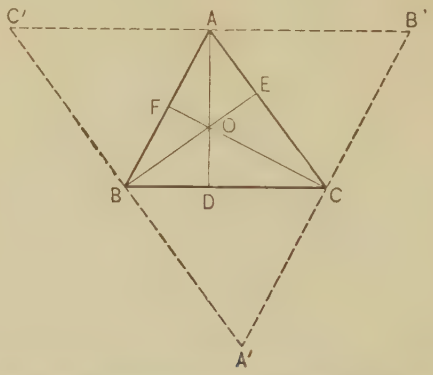
Ex. 2. *Construct a circle which passes through the three vertices of a given triangle.*

Ex. 3. Prove that only *one* circle can be passed through the three vertices of a triangle.

Ex. 4. Prove that the center of the circle about an equilateral $\triangle$ lies in each of its $\angle$ bisectors.

Proposition XLVII. Theorem

163. *The three altitudes of a triangle (produced if necessary) meet in a point.*



Given $\triangle ABC$ and the altitudes AD, BE, and CF.

To prove AD, BE, and CF meet in a common point.

Proof. Through the vertices of ABC, draw $B'C' \parallel BC$, $A'C' \parallel AC$, and $A'B' \parallel AB$. (Why?)

Since $AD \perp BC$, (Hyp.)

$AD \perp B'C'$. (105)

But $ABCB'$ and $ACBC'$ are parallelograms. (136)

$\therefore AB' = BC$, and $AC' = BC$. (139)

$\therefore AB' = AC'$. (Ax. 1.)

Or AD is the $\perp$ bisector of $B'C'$.

In like manner it follows that

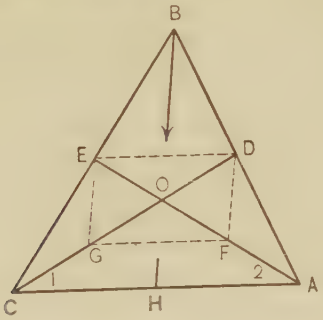
BE is the $\perp$ bisector of $A'C'$,

and CF is the $\perp$ bisector of $A'B'$.

$\therefore AD$, BE, and CF meet in a point (*i.e.* are concurrent). (162)

Proposition XLVIII. Theorem

164. *The medians of a triangle intersect in a point which is two thirds of the distance from a vertex to the mid-point of the opposite side.*



Given $\triangle ABC$ with medians AE, CD, BH.

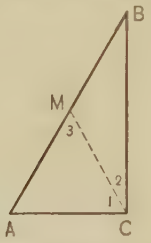
To prove AE, CD, and BH meet in a point O, and $AO = \frac{2}{3}\,AE$, $CO = \frac{2}{3}\,CD$, $BO = \frac{2}{3}\,BH$.

Proof

STATEMENTS	REASONS
1. Let $CG = GO$ and $OF = FA$.	1. (81)
2. Draw ED, DF, FG, and GE.	2. Why?
3. $ED = \frac{1}{2}\,CA$ and $ED \parallel CA$.	3. (153)
4. $GF = \frac{1}{2}\,CA$ and $GF \parallel CA$.	4. (153)
5. $ED = GF$ and $ED \parallel GF$.	5. Why?
6. $EGFD$ is a parallelogram.	6. (144)
7. $OE = OF$ and $OD = OG$.	7. (146)
8. Hence $OE = OF = FA$ and $OD = OG = CG$.	8. Why?
9. $AO = \frac{2}{3}\,AE$, and $CO = \frac{2}{3}\,CD$.	9. Why?
10. Likewise BH passes through O and $BO = \frac{2}{3}\,BH$.	10. Steps (1–9)

Proposition XLIX. Theorem

165. *If one acute angle of a right triangle is double the other, the hypotenuse is double the shorter side (arm).*



Given right $\triangle ABC$, right $\angle$ at C, and $\angle A = 2\angle B$.
To prove $AB = 2\,AC$.

Proof

STATEMENTS	REASONS
1. $\angle A = 2\angle B$, $\angle C = 90°$.	1. Hyp.
2. $\angle A + \angle B = 90°$.	2. Why?
3. $\therefore$ $\angle A = 60°$, $\angle B = 30°$.	3. Why?
4. Draw CM making $\angle 1 = \angle A = 60°$.	4. Why?
5. $\angle 3 = 60°$.	5. Why?
6. $\angle 2 = 30°$.	6. Why?
7. $AM = AC = MC$.	7. Why?
8. $MB = MC$.	8. Why?
9. $\therefore AB = AM + MB = 2\,AC$.	9. Why?

Ex. 1. If in triangle ABC, $\angle A = 60°$, $\angle B = 30°$, and $AB = 12$ inches, find the length of AC.

Ex. 2. If in triangle ABC, $AB = 10$ inches, $\angle A = 60°$, $\angle C = 90°$, and CD is the altitude upon AB, find the length of AD.

Ex. 3. The base angles A and B of isosceles triangle ABC equal 15°, and $BC = 8$ inches. Find the length of the altitude AD.

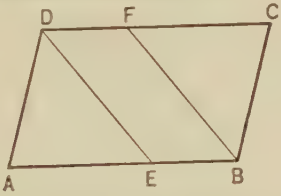
Ex. 4. In quadrilateral $ABCD$, $AB = BC = 6$ inches, $\angle A = 120°$, $\angle B = 60°$, and $\angle C = 150°$. Find the length of AD.

Ex. 5. DC, the upper base of trapezoid $ABCD$, equals 14 inches, and $AB = BC$. Find the length of AB, if $\angle A = 90°$, and $\angle B = 60°$.

ANALYSIS OF THEOREMS

166. The demonstration of a theorem is most frequently discovered by the so-called **analytic method**. An analysis considers the various "*means*" by which a conclusion can be proved, and thereby reduces the original proposition to another, simpler one. This reduction is continued until a known fact is obtained. A concrete example will explain the true character of this method.

167. THEOREM. *The bisectors of a pair of opposite angles of an oblique parallelogram are parallel.*



Given DE and BF, the bisectors of two angles of the oblique parallelogram $ABCD$.

To prove $DE \parallel BF$.

Analysis. 1. The means for proving that two lines are parallel are:

(*a*) A parallelogram.
(*b*) Equal alternate interior angles.
(*c*) Corresponding angles.
(*d*) Supplementary interior angles, etc.

Let us select any one of these methods, *e.g.* 1 (*a*), *i.e.*

to prove $EBFD$ is a parallelogram.

2. The means for proving that a quadrilateral is a $\square$ are:

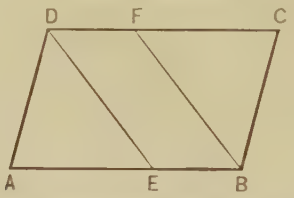
(*a*) Opposite sides are equal.
(*b*) Two opposite sides are equal and parallel, etc.

Again we select any one, *e.g.* 2 (*a*), *i.e.*

to prove $ED = BF$, $EB = DF$.

3. The means for proving the equality of lines is usually a pair of congruent triangles, *i.e.*

to prove $\triangle AED \cong \triangle BCF$.



4. The congruence of the two triangles is easily established. Hence we obtain the following demonstration:

Proof.

	$AD = BC.$	(139)
	$\angle A = \angle C.$	(139)
	$\angle ADE = \angle FBC.$	(Ax. 8.)
	$\therefore \triangle ADE \cong \triangle FBC.$	(a. s. a. = a. s. a.)
	$\therefore ED = BF$, and $AE = FC.$	(corr. parts, etc.)
But	$AB = DC.$	(139)
	$\therefore DF = EB.$	(Ax. 3.)
	$\therefore DEBF$ is a $\square$.	(143)
	$\therefore ED \parallel BF.$	(136)

The above demonstration is not the only one, nor the shortest one. Each of the possibilities indicated under 1 (*a*), 1 (*b*), 1 (*c*), 1 (*d*), 2 (*a*), 2(*b*), will furnish one or more proofs.

Ex. **1.** Prove the above proposition by means of 2 (*b*).

Ex. **2.** Prove the above proposition by means of 1 (*b*).

Ex. **3.** Prove the above proposition by means of 1 (*c*).

168. When in an analysis new lines or angles are drawn, the student should — before proceeding with the regular analysis — determine whether thereby any new equalities of lines or of angles are produced.

169. METHOD XI. **Geometrical propositions may be demonstrated by algebraical computation.**

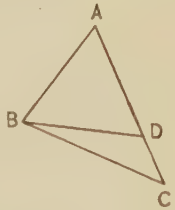
At this stage of the work, the above method may be employed to establish relations between two or more angles. In case of two angles, express the value of one in algebraic form (*e.g.* $m°$) and from this try to derive the value of the other.

If a relation between three or more angles has to be demonstrated, express the values of all of them excepting one by algebraic symbols (as $m°$, $n°$, $p°$, etc.), and try to discover the value of the remaining angle. After some practice the symbols which designate the angle, as $\angle A$ or $\angle B$, may be used to express their value.

If the student is unable to find the value of the last angle when all others are given in algebraic form, he should at first attack the problem for numerical values of these angles and then try to solve the general problem.

Ex. Given $\triangle ABC$, and point D in AC, so that $AD = AB$.

Prove $\angle DBC = \frac{1}{2}(\angle B - \angle C)$.



Proof. Let $\angle B = m°$, and $\angle C = n°$.

Then $\angle A = 180° - m° - n°$.

But $\angle ABD + \angle BDA = 180° - \angle A$,

i.e. $\angle ABD + \angle BDA = 180° - (180° - m° - n°)$

$= m° + n°$.

But $\angle ABD = \angle ADB$.

$\therefore 2\angle ABD = m° + n°$.

$\therefore \angle ABD = \frac{m°}{2} + \frac{n°}{2}$.

But $\angle DBC = \angle B - \angle ABD$

$= m° - \left(\frac{m°}{2} + \frac{n°}{2}\right)$

$= \frac{m°}{2} - \frac{n°}{2}$

$= \frac{\angle B}{2} - \frac{\angle C}{2}$

or $\angle DBC = \frac{1}{2}(\angle B - \angle C)$.

Ex. 1. An exterior angle formed by the prolongation of the base of an isosceles triangle equals 90° increased by one half the vertex angle.

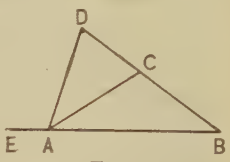
Ex. 2. The sum of the two exterior angles of a right triangle which are formed by the prolongations of the hypotenuse equals 270°.

Ex. 3

Ex. 3. If in the base AC of an isosceles $\triangle ABC$, two points D and E be taken such that

$$AE = AB \text{ and } CD = BC,$$

prove $$\angle DBE = \angle A.$$



Ex. 4

Ex. 4. If $AD = AC = CB$, and DB and EB are st. lines, then

$$\angle EAD = 3\angle B.$$

Ex. 5. If the arm CA of isosceles triangle ABC be produced so that AD equals the base AB, then

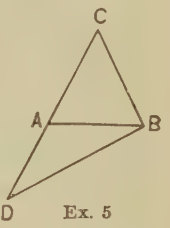
$$\angle C = 180° - 4\angle D.$$

Ex. 6. If the base AB of isosceles triangle ABC is produced to D, and CD is drawn, then

$$\angle BCD = \angle A - \angle D.$$

Ex. 7. If in isosceles triangle ABC, one end B of the base AB is connected with a point D in the opposite arm, then

$$\angle A = \tfrac{1}{2}(\angle CDB + \angle CBD).$$



Ex. 5

Ex. 8. If the side AC of triangle ABC is produced to D so that $CD = CB$, then

$$\angle ABD = 90° + \tfrac{1}{2}(\angle ABC - \angle A).$$

Ex. 9. If AD is the altitude and AE the bisector of the angle BAC of the triangle ABC, prove

$$\angle DAE = \tfrac{1}{2}(\angle B - \angle C).$$

Ex. 9

Ex. 10. The sum of three angles of a quadrilateral, diminished by the fourth exterior angle, is equal to a straight angle.

170. In proving an exercise, try to get as neat and as short a proof as possible.

MISCELLANEOUS EXERCISES

Ex. 1. What is the angle between the hands of a clock at 12 min. past 11? At 6:30? At 8:45?

Ex. 2. The vertex angle of an isosceles triangle is 18° 44′. How many degrees are there in each base angle?

Ex. 3. An exterior $\angle$ of an equiangular polygon equals one fifth of a right angle. Find the number of sides of the polygon.

Ex. 4. Prove: *The line perpendicular to an arm of a right triangle at its middle point passes through the mid-point of the hypotenuse.*

Ex. 5. Prove: *In any right triangle the median to the hypotenuse is one half the hypotenuse.* (Use Ex. 4.)

Ex. 6. In $\triangle ABC$, $\angle A = 60°$, $\angle B > \angle C$. Which is the greatest side?

Ex. 7. Construct a quadrilateral three of whose angles are 150°, 90°, and 60°.

Ex. 8. What are some of the special properties of an isosceles triangle? Equilateral triangle? Rhombus? Rectangle? Square?

Ex. 9. Construct $\triangle ABC$, given $AB = 3''$, $\angle A = 60°$, $\angle B = 45°$. Then construct the altitude from A to BC, the median from B to AC, and the bisector of $\angle C$.

Ex. 10. What are the different tests by which to determine the congruence of triangles? The equality of angles? The equality of lines? The parallelism of lines? The perpendicularity of lines?

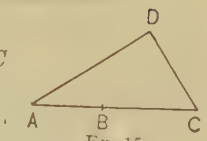
Ex. 11. What are the different tests for determining the inequality of two lines? Of two angles?

Ex. 12. *The bisectors of two supplementary adjacent angles are perpendicular to each other.*

Ex. 13. If the bisectors of two adjacent angles are perpendicular to each other, the exterior sides of these angles form a straight line.

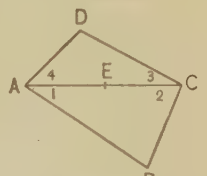
Ex. **14**. *The bisectors of vertical angles are in a straight line.*

Ex. **15**. If $\angle A + \angle C + \angle D = 180°$, then ABC is a straight line.



Ex. 15

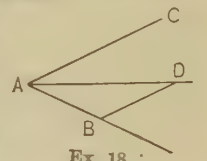
Ex. **16**. If $\angle 1 + \angle 2 + \angle B = \angle 3 + \angle 4 + \angle D$, then AEC is a straight line.



Ex. 16

Ex. **17**. *The altitude upon the hypotenuse of a right triangle divides the figure into two triangles which are mutually equiangular.*

Ex. **18**. If through any point D on the bisector of an angle A, a parallel be drawn to one of the sides to meet the other side in B, then $AB = BD$.



Ex. 18

Ex. **19**. If from a point in the bisector of an angle, lines are drawn parallel to the sides of the angle, a rhombus is formed.

Ex. **20**. If the vertex angles of two isosceles triangles are supplementary, the base angles are complementary.

Ex. **21**. If an arm of an isosceles triangle is produced by its own length through the vertex, and the end of the prolongation is joined to the nearest end of the base, the line joining is perpendicular to the base.

Ex. **22**. *Corresponding medians of congruent triangles are equal.*

Ex. **23**. *Corresponding altitudes of congruent triangles are equal.*

Ex. **24**. *Corresponding angle-bisectors of congruent triangles are equal.*

Ex. **25**. Two isosceles triangles are congruent if the vertex angle and the altitude upon an arm of the one are respectively equal to the vertex angle and the corresponding altitude of the other.

Ex. **26**. If in the pentagon $ABCDE$,

$$AB = BC,\ AE = CD, \text{ and } \angle A = \angle C,$$

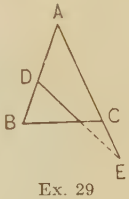
then

$$BE = BD, \text{ and } \angle E = \angle D.$$

Ex. **27**. *Two equilateral triangles are congruent if the altitude of one equals the altitude of the other.*

Ex. 28

Ex. 28. If the opposite sides of a hexagon are parallel, and two of the opposite sides are equal, all opposite sides are equal.

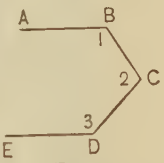


Ex. 29

Ex. 29. If from the ends of the base BC of an isosceles triangle ABC, equal parts, BD and CE, be laid off on one arm and the prolongation of the other, the line joining D and E is bisected by the base.

Ex. 30. If two lines are intersected by a transversal, and the bisectors of the interior angles on the same side of the transversal are perpendicular to each other, these lines are parallel.

Ex. 31. *If the opposite angles of a quadrilateral are equal, the figure is a parallelogram.*



Ex. 32

Ex. 32. If $AB \parallel ED$, $\angle 1 + \angle 2 + \angle 3 = 4$ rt. $\angle$s.

Ex. 33. State and prove the converse of the preceding proposition.

Ex. 34. Find the number of diagonals in a polygon of 5 sides; of 8 sides; of 10 sides; of n sides.

Ex. 35. How many sides has a polygon, the sum of whose interior angles equals three times the sum of the exterior angles? (*i.e.* one ext. $\angle$ at each vertex.)

Ex. 36. How many sides has a polygon the sum of whose interior angles is equal to the sum of the exterior angles?

Ex. 37. How many sides has a polygon, the sum of whose interior angles is equal to three times the sum of the angles of a hexagon?

Ex. 38. How many sides has an equiangular polygon whose exterior angle equals the interior angle of an equilateral triangle?

Ex. 39. If the upper base of an isosceles trapezoid is equal to one of the arms, the diagonals bisect the angles at the lower base.

Ex. 40. If a perpendicular be dropped from the vertex to the base of a triangle, each segment of the base will be shorter than the adjacent side of the triangle.

Ex. 41. If the vertices of a triangle lie in the sides of another triangle, the perimeter of the first is less than the perimeter of the second.

Ex. 42. The perpendiculars from two vertices of a triangle upon the median drawn from the third vertex are equal.

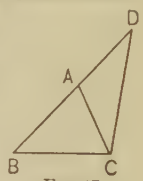
Ex. 43. The lines joining the mid-points of the sides of a rectangle, taken in order, form a rhombus.

Ex. 44. The lines joining the mid-points of opposite sides of any quadrilateral, taken in order, form a parallelogram.

* Ex. 45. The mid-points of two opposite sides of a quadrilateral and the mid-points of the diagonals determine the vertices of a parallelogram.

Ex. 46. A line from the vertex of an isosceles triangle to any point in the base is shorter than either arm.

Ex. 47. If in the triangle ABC, $AB > AC$, and D is a point in the prolongation of BA, then $DB > DC$.



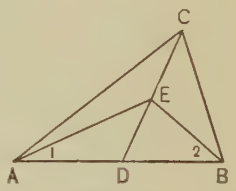
Ex. 47

Ex. 48. Lines joining the mid-points of two opposite sides of a parallelogram to the ends of a diagonal trisect the other diagonal.

Ex. 48

* Ex. 49. If a point D in a side BC of triangle ABC is joined to A, and $AC = BC$, $AB = AD = DC$, then $\angle C = 36°$.

* Ex. 50. If any point E in the median CE is joined to A and B and $\angle B > \angle A$, prove that $\angle 2 > \angle 1$.



Ex. 50

Ex. 51. *From a given point without a line, to draw a line forming with the given line an angle equal to half a right angle.*

Ex. 52. From a given point without a line, to draw a line forming with the given line an angle of 60°. How many such lines can be drawn?

Ex. 53. From a given point without a line to draw a line, making a given angle with the given line.

Ex. 54. Construct a line terminating in the sides of a given angle and equal and parallel to a given line.

Ex. 55. *The diagonals of an isosceles trapezoid are equal.*

* Ex. 56. If the diagonals of a trapezoid are equal, the trapezoid is isosceles.

171. METHOD XII. **In order to prove that the sum of two lines, a and b, equals a third line, c, either**

(a) Construct the sum of a and b, and prove the line so obtained is equal to c, or

(b) Lay off a (or b) on c, and prove that the line representing the difference equals b (or a).

Ex. 57

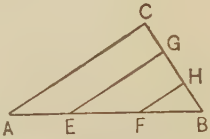
Ex. 57. The sum of the perpendiculars dropped from any point in the base of an isosceles triangle to the arms is equal to the altitude upon one of the arms.

Ex. 58. In $\triangle ABC$, if

$$AE = BF$$

and $AC \parallel EG \parallel FH$,

prove that $EG + FH = AC$.

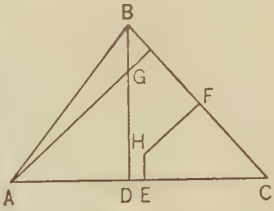


Ex. 58

Ex. 59. If through a point D in the base AB of isosceles triangle ABC, parallels are drawn to the arms meeting the arms in E and F respectively, then

$$DF + DE = AC.$$

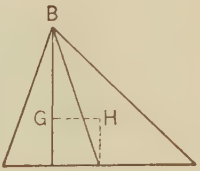
Ex. 60. The sum of the three perpendiculars dropped from any point within an equilateral triangle upon the sides is constant, and equal to the altitude of the triangle. (Ex. 57.)



Ex. 61

* **Ex. 61.** If the altitude BD of $\triangle ABC$ is intersected by another altitude in G, and EH and HF are perpendicular-bisectors, prove

$$BG = 2(HE), \text{ and } AG = 2(HF).$$



Ex. 62

Ex. 62. The line joining the point of intersection of the altitudes of a triangle and the point of intersection of the three perpendicular-bisectors cuts off one third of the corresponding median. (Ex. 61.)

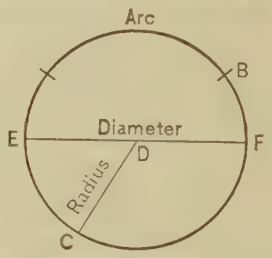
* **Ex. 63.** The points of intersection of the altitudes, medians, and perpendicular-bisectors of a triangle lie in a straight line.

BOOK II

THE CIRCLE — CONSTRUCTIONS

172. DEF. A **circle** is a plane closed curve, having all its points equally distant from a fixed point in the plane, as *ABC*(37).

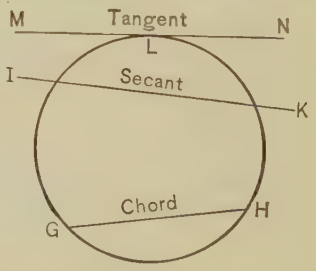
173. DEF. The fixed point (*D*) is the **center**. The length of the circle is called the **circumference**. An **arc** is any portion of the circle, as *AB*. A **semicircle** is half of the circle. A **minor arc** is an arc less than a semicircle; a **major arc** is an arc greater than a semicircle.



The word *arc* taken alone generally signifies a minor arc.

174. DEF. A **radius** is a line joining the center to any point in the circle, as *DC*. A **diameter** is a straight line through the center terminated at each end by the circle, as *EF*.

175. DEF. A **chord** is a straight line joining any two points in the circle, as *GH*. A **secant** is a straight line that *intersects* the circle in two points, as *IK*. A **tangent** is a straight line that *touches* the circle at one point only, and does not intersect it if produced, as *MN*. This point (*L*) is called the *point of contact* or *point of tangency*.



176. DEF. A **central angle** is an angle formed by two radii, as $\angle EDC$.

An angle is said to *intercept* any arc that is cut off by its sides, and this arc is said to *subtend* the angle. A chord that joins the ends of an arc is said to *subtend* this arc; thus chord GH subtends the arc GH.

177. DEF. Circles having the *same center* are called **concentric** circles.

PRELIMINARY THEOREMS

178. *All radii of the same circle are equal.* (By definition.)

179. *A point is within a circle if its distance from the center is less than a radius. A point is on (or in) the circle if its distance from the center is equal to a radius. A point is without a circle if its distance from the center is greater than a radius.*

180. *Two circles are equal if their radii are equal.* (Prove by superposition.)

181. *All diameters of the same or of equal circles are equal.*

182. *A diameter bisects the circle.* (Prove by superposition.)

NOTE: 1. Diameters are *central* chords. All other chords are *non-central* chords.

2. A circle is determined by its center and a radius.

Ex. 1. Construct a circle with center A and radius equal to r.

Ex. 2. Construct a central angle of 90°, 60°, 45°, 30°.

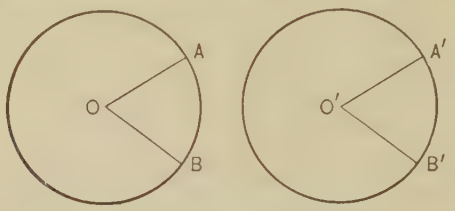
Ex. 3. Prove: Every diameter bisects the circle.

Ex. 4. Prove: A circle cannot have two centers.

Ex. 5. Construct two concentric circles, then construct a circle concentric to the first two circles and midway between them.

Proposition I. Theorem

183. *In the same circle or in equal circles, equal central angles have equal arcs; and, conversely, equal arcs have equal central angles.*



I. **Given** in equal Ⓢ, O and O', $\angle O = \angle O'$.

To prove $\overset{\frown}{AB} = \overset{\frown}{A'B'}$.

Proof

STATEMENTS	REASONS
1. Place circle O on its equal circle O' so that OB coincides with $O'B'$.	1. By hyp., the Ⓢ are =. Axiom 19.
2. OA falls along $O'A'$.	2. By hyp., $\angle O = \angle O'$.
3. A coincides with A'.	3. Radii of = Ⓢ are =.
4. $\overset{\frown}{AB}$ coincides with $\overset{\frown}{A'B'}$.	4. Radii of = Ⓢ are =.
5. $\therefore \overset{\frown}{AB} = \overset{\frown}{A'B'}$.	5. (19).

II. Conversely. **Given** in equal Ⓢ, O and O', $\overset{\frown}{AB} = \overset{\frown}{A'B'}$.

To prove $\angle O = \angle O'$.

1. Statement (1) above.	1. By hyp., the Ⓢ are =.
2. $\overset{\frown}{BA}$ falls along $\overset{\frown}{B'A'}$.	2. Equal Ⓢ have = radii.
3. Point A falls on A'.	3. By hyp., $\overset{\frown}{AB} = \overset{\frown}{A'B'}$.
4. OA coincides with $O'A'$.	4. Two pts. det. a line.
5. $\therefore \angle O = \angle O'$.	5. (19).

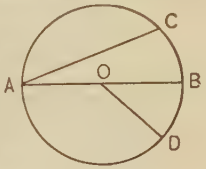
184. Cor. *In the same or in equal circles, the greater of two unequal central angles intercepts the greater arc, and conversely.* Hint. By superposition.

[Practical applications, pages 287–290.]

185. Method XIII. **The equality of arcs in the same circle or equal circles can be demonstrated by the equality of their central angles.**

Ex. 1. If AB and CD are diameters of the same circle, arc AC = arc BD.

Ex. 2. If AB is a diameter, OD a radius, AC a chord, and $\angle BOD = 2\angle A$, then $\overset{\frown}{BD} = \overset{\frown}{BC}$.



Ex. 2

Ex. 3. If a diameter AB bisects the angle A, which is formed by two chords AC and AD, then $\overset{\frown}{BC} = \overset{\frown}{BD}$.

Ex. 4. If from a point A in a circle a chord AB and a diameter AC are drawn, a radius parallel to AB bisects $\overset{\frown}{BC}$.

Ex. 5. If through a point equidistant from two points in a circle a radius is drawn, the arc between the two points is bisected.

Ex. 6. If the perpendiculars drawn from a point in the circle upon two radii are equal, the point bisects the arc intercepted by the two radii.

Ex. 7. Divide a circle into four equal parts; into 8; into 6.

186. A circle may be divided into 360 equal parts called *degrees*. A degree may be subdivided into 60 equal parts called *minutes*, and similarly a minute may be divided into 60 *seconds*.

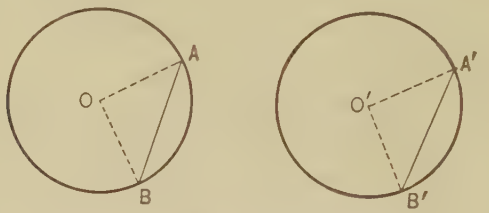
An angle is *measured by* an arc, if both angle and arc have the same number of degrees.

Ex. 8. Prove that a central angle is measured by its intercepted arc if the central angle equals: (*a*) 90°; (*b*) 30°; (*c*) 1°.

Ex. 9. Construct an arc of (*a*) 45°, (*b*) 60°, (*c*) 150°.

Proposition II. Theorem

187. *In the same circle, or in equal circles, equal arcs have equal chords; and, conversely, equal chords have equal arcs.*



I. **Given** in equal ⓢ, O and O', $\overset{\frown}{AB} = \overset{\frown}{A'B'}$.
To prove $AB = A'B'$.

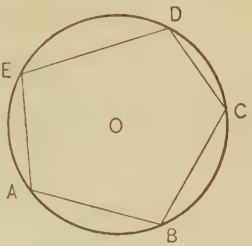
Proof

STATEMENTS	REASONS
1. Draw $OA, OB, O'A', O'B'$.	1. Ax. 13.
2. $\overset{\frown}{AB} = \overset{\frown}{A'B'}$	2. Hyp.
3. $\angle AOB = \angle A'O'B'$.	3. In equal ⓢ, equal arcs have equal central ∡.
4. $AO = A'O'$, $BO = B'O'$.	4. Radii of equal ⓢ are =.
5. $\triangle ABO \cong \triangle A'B'O'$.	5. s. a. s. = s. a. s.
6. $\therefore AB = A'B'$.	6. Corr. sides of congruent △ are =.

II. Converse. **Given** in equal ⓢ, O and O', $AB = A'B'$.
To prove $\overset{\frown}{AB} = \overset{\frown}{A'B'}$.

1. Draw $OA, OB, O'A', O'B'$.	1. A st. line etc.
2. $AB = A'B'$.	2. Hyp.
3. $AO = A'O'$, $BO = B'O'$.	3. Radii of = ⓢ are =.
4. $\triangle ABO \cong \triangle A'B'O'$.	4. s. s. s. = s. s. s.
5. $\angle AOB = \angle A'O'B'$.	5. Corr. ∡ of ≅ △ are =.
6. $\therefore \overset{\frown}{AB} = \overset{\frown}{A'B'}$.	6. In = ⓢ, equal central ∡ have = arcs.

188. DEF. A polygon is **inscribed in a circle**, if all its vertices are in the circle, as $ABCDE$. The circle is then said to be **circumscribed about the polygon.**

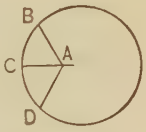


The center of the circumscribed circle is called the **circumcenter** of the polygon.

189. METHOD XIV. **The equality of arcs, in the same circle or equal circles, may also be demonstrated by the equality of their chords, and vice versa.**

Ex. **1.** If $\angle BAC = \angle DAC$ and $AB = AD$, then $\overset{\frown}{BC} = \overset{\frown}{CD}$.

Ex. **2.** If $\angle BAC = \angle DAC$ and $\angle CBA = \angle CDA$ (not drawn in the figure), then $\overset{\frown}{CB} = \overset{\frown}{CD}$.



Exs. 1, 2

Ex. **3.** If $\triangle ABC$ is inscribed in a circle and $\angle A = \angle B$, prove that $\overset{\frown}{AC} = \overset{\frown}{BC}$.

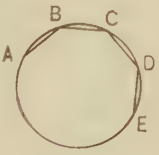
Ex. **4.** If two chords bisect each other, the arcs intercepted by the sides of a pair of vertical angles are equal.

Ex. **5.** *If two chords bisect each other, they are diameters*

Ex. **6.** If chords AB, BC, CD, DE are equal, then chords AC, BD, CE are equal.

Ex. **7.** If $AB = CD$, then $BC = AD$.

Ex. **8.** If two intersecting chords, AD and BC, are equal, then $AB = CD$.



Ex. 6

Exs. 7, 8

Ex. **9.** The diagonals of an equilateral pentagon inscribed in a circle are equal.

Ex. **10.** The radii drawn to the vertices of an inscribed equilateral hexagon divide the figure into six congruent equilateral triangles.

Proposition III. Theorem

190. *The diameter perpendicular to a chord bisects the chord and the arcs of the chord.*

Given in $\odot O$, the diam. $FD \perp AB$.

To prove $AE = EB$, $\overset{\frown}{AD} = \overset{\frown}{DB}$, $\overset{\frown}{AF} = \overset{\frown}{FB}$.

Proof

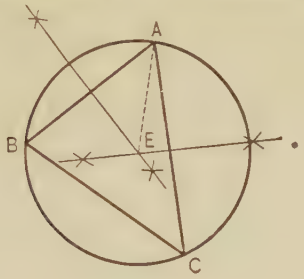
STATEMENTS	REASONS
1. Draw OA, OB.	1. Ax. 13.
2. $OE \perp AB$.	2. Hyp.
3. $\angle 1 = \angle 2 =$ rt. $\angle$.	3. $\perp$s form right $\measuredangle$.
4. $OA = OB$.	4. Radii of a $\odot$ are $=$.
5. $OE = OE$.	5. Iden.
6. Rt. $\triangle AOE \cong$ rt. $\triangle BOE$.	6. hy. a. = hy. a.
7. $\therefore AE = EB$, $\angle 3 = \angle 4$.	7. Corr. pts. of $\cong$ $\triangle$ are $=$.
8. $\therefore \overset{\frown}{AD} = \overset{\frown}{DB}$.	8. Equal cent. $\measuredangle$ have $=$ arcs.
9. $\overset{\frown}{DAF} = \overset{\frown}{DBF}$.	9. A diameter bisects the $\odot$.
10. $\therefore \overset{\frown}{AF} = \overset{\frown}{FB}$.	10. Equals from equals, rem. $=$.

191. Cor. *The perpendicular-bisector of a chord passes through the center of the circle.*

Ex. If in circle O, $AB = AC$, $OD \perp AB$, and $OE \perp AC$, prove that $\triangle ADO \cong \triangle AEO$.

PROPOSITION IV. PROBLEM

192. *To circumscribe a circle about a given triangle.*



Given $\triangle ABC$.

Required to circumscribe a circle about $\triangle ABC$.

Construction. Draw the perpendicular-bisectors of the sides AC and AB. They will intersect at some point E. (82)(102)

From E as a center, with a radius equal to EA, describe a circle ABC.

ABC is the required circle.

Proof. E is equidistant from A, B, and C. (162)

$\therefore$ A circle drawn from E as a center with a radius equal to EA will pass through A, B, and C.

193. COR. 1. *Three points not in a straight line determine a circle.*

194. COR. 2. *A circle cannot be drawn through three points which lie in the same straight line.*

195. COR. 3. *A straight line cannot intersect a circle in more than two points.*

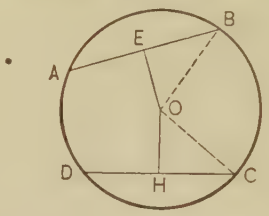
196. COR. 4. *Two circles cannot meet in more than two points.*

EX. **1.** *To find the center of a given circle.*

EX. **2.** *To bisect a given arc.*

Proposition V. Theorem

197. *In the same circle, or in equal circles, equal chords are equally distant from the center; and, conversely, chords equally distant from the center are equal.*



I. **Given** in $\odot ABCD$, $AB = CD$, $OE \perp AB$, $OH \perp CD$.

To prove $OE = OH$.

Proof

STATEMENTS	REASONS
1. Draw OB and OC.	1. A st. line may be drawn between two pts.
2. $AB = CD$, $OE \perp AB$, $OH \perp DC$.	2. Hyp.
3. E and H are mid-pts. of the chords.	3. Dia. $\perp$ chord bisects the chord.
4. $EB = HC$.	4. Halves of equals are $=$
5. $OB = OC$.	5. Radii of a $\odot$ are $=$.
6. $\angle BEO = \angle OHC$ = a rt. $\angle$.	6. $\perp$s form rt. $\measuredangle$.
7. Rt. $\triangle OEB \cong$ rt. $\triangle OHC$.	7. hy. a = hy. a.
8. $\therefore OE = OH$.	8. Corr. sides of $\cong$ $\triangle$ are $=$

Ex. 1. The perpendiculars from the center to the sides of an inscribed equilateral polygon are equal.

Ex. 2. Prove proposition V by drawing AO and DO and then by proving $\triangle AOB \cong \triangle DOC$.

Hint. See p. 98, Ex. 23.

II. Conversely. **Given** in $\odot ABCD$,
$OE = OH$, $OE \perp AB$, $OH \perp CD$.

To prove $AB = CD$.

1. Draw OB and OC.	1. Ax. 13.
2. $OB = OC$.	2. Radii same $\odot$ are $=$.
3. $OE = OH$, $OE \perp AB$, $OH \perp CD$.	3. Hyp.
4. $\angle BEO = \angle OHC =$ rt. $\angle$	4. $\perp$s form rt. $\angle$s.
5. Rt. $\triangle OEB \cong$ rt. $\triangle OHC$.	5. hy. a. = hy. a.
6. $EB = HC$.	6. Corr. sides $\cong$ $\triangle$s are $=$.
7. But E and H are midpts. of the chords.	7. Prop. III.
8. $\therefore AB = CD$.	8. Why?

198. Method XV. **The equality of two chords in the same circle, or in equal circles, is usually established by means of equal distances from the center or equal subtended arcs.**

Ex. 1. If through any point in a radius two chords are drawn making equal angles with the radius, these chords are equal.

Ex. 2. A line joining the point of intersection of two equal chords to the center bisects the angle formed by the chords.

Ex. 3. In a given circle, to draw a chord equal and parallel to a given chord.

Ex. 4. In a given circle, to draw a chord equal to a given chord, and parallel to a given line.

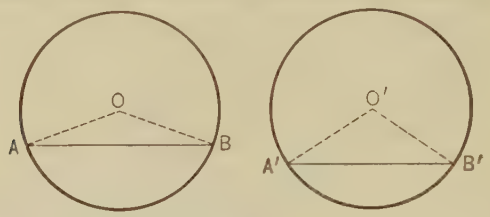
Ex. 5. At the ends of a diameter, chords are drawn making equal angles with the diameter. Prove they are equal.

Ex. 6. A diameter bisects each of two non-central chords. Show the chords are parallel.

Ex. 7. Two intersecting chords have a segment of each equal. Prove the chords equal.

Proposition VI. Theorem

199. *In the same circle, or in equal circles, the greater of two minor arcs has the greater chord; and conversely, the greater chord has the greater arc.*



I. **Given** in equal Ⓢ, O and O',

$$\overset{\frown}{AB} > \overset{\frown}{A'B'}.$$

To prove

$$AB > A'B'.$$

Proof

STATEMENTS	REASONS
1. Draw OA, OB, $O'A'$, $O'B'$.	**1.** Why?
2. $\overset{\frown}{AB} > \overset{\frown}{A'B'}$.	**2.** Hyp.
3. In $\triangle AOB$ and $A'O'B'$, $\angle O > \angle O'$.	**3.** (184).
4. $AO = A'O'$ and $BO = B'O'$.	**4.** Why?
5. $\therefore AB > A'B'$.	**5.** (132).

II. Conversely **Given** in equal Ⓢ, O and O',

$$AB > A'B'.$$

To prove

$$\overset{\frown}{AB} > \overset{\frown}{A'B'}.$$

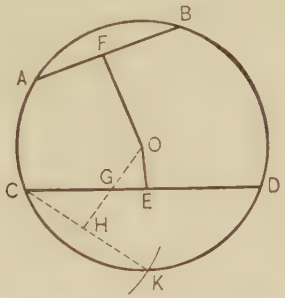
1. Draw OA, OB, $O'A'$, $O'B'$.	**1.** Why?
2. $AB > A'B'$.	**2.** Hyp.
3. $OA = O'A'$ and $OB = O'B'$.	**3.** Why?
4. In $\triangle AOB$ and $A'O'B'$, $\angle O > \angle O'$.	**4.** (133).
5. $\therefore \overset{\frown}{AB} > \overset{\frown}{A'B'}$.	**5.** (184).

Note. Not all proofs will be complete hereafter. The student must supply the parts wanting.

[See p. 115 for exercises.]

Proposition VII. Theorem

200. *In the same circle, or in equal circles, if two chords are unequal, the greater chord is at the less distance from the center.*



Given in $\odot ABDC$, $CD > AB$, $OE \perp CD$, and $OF \perp AB$.

To prove $OE < OF$.

Proof

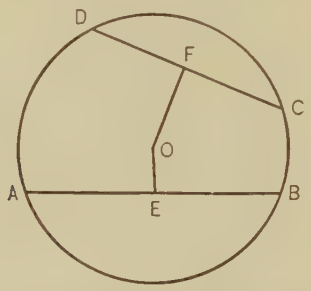
STATEMENTS	REASONS
1. $CD > AB$.	**1.** Hyp.
2. $\overset{\frown}{CD} > \overset{\frown}{AB}$.	**2.** (199).
3. From C as center, and radius equal to AB, strike off an arc to cut $\overset{\frown}{CD}$ within at K.	**3.** Why?
4. Draw CK, and $OH \perp CK$.	**4.** Why?
5. $CK = AB$.	**5.** Why?
6. $OH = OF$.	**6.** Equal chords are = dist. from center.
7. $OE < OG$.	**7.** (131).
8. $OG < OH$.	**8.** (Ax. 11).
9. $\therefore OE < OH$ or $OE < OF$.	**9.** (Ax. 9).

[See p. 115 for exercises.]

Proposition VIII. Theorem

201. *In the same circle, or in equal circles, if two chords are unequally distant from the center, the chord at the less distance is the greater.*

[Converse to Prop. VII.]



Given in $\odot ABCD$ with $OE < OF$, $OE \perp AB$, $OF \perp DC$.

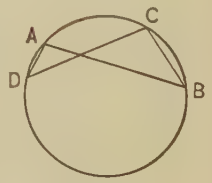
To prove $AB > DC$.

Proof

STATEMENTS	REASONS
1. Either $AB < DC$, or $AB = DC$, or $AB > DC$.	**1.** Why?
2. If $AB < DC$, then $OE > OF$.	**2.** (200).
3. If $AB = DC$, then $OE = OF$.	**3.** (197).
4. Statements 2, 3 are false.	**4.** By hyp., $OE < OF$.
5. $\therefore AB > DC$.	**5.** Why?

Ex. 1. If $AB > CD$, and $\overset{\frown}{ACB}$ and $\overset{\frown}{DAC}$ are minor arcs, prove that $CB > AD$.

Ex. 2. If $CB > AD$, and $\overset{\frown}{DAC}$ and $\overset{\frown}{ACB}$ are minor arcs, prove that $AB > CD$.



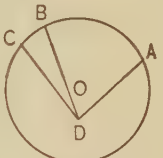
Exs. 1, 2

202. METHOD XVI. **The inequality of chords, in the same circle or in equal circles, is usually established by means of unequal distances from the center or by means of unequal arcs.**

Ex. 1. If ABC is any triangle inscribed in a circle and $\angle A > \angle B$, then $\overset{\frown}{BC} > \overset{\frown}{AC}$.

Ex. 2. State and prove the converse of Exercise 1.

Ex. 3. If $AD = DC$,
and $\angle ADB > \angle BDC$,
then $\overset{\frown}{AB} > \overset{\frown}{BC}$.



Exs. 3, 4

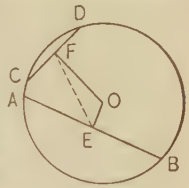
Ex. 4. If $AD = DC$, and $\overset{\frown}{AB} > \overset{\frown}{BC}$, then $\angle ADB > \angle BDC$.

Ex. 5. If upon a radius two perpendicular chords are drawn, the one nearer the center is the greater chord.

Ex. 6. The perpendicular from the center of a circle to a side of an inscribed equilateral hexagon is less than the perpendicular from the center to the side of an inscribed equilateral octagon.

Ex. 7. If $EO \perp AB$, $OF \perp CD$, and $\angle OEF > \angle OFE$, then $AB > CD$.

Ex. 8. If $\triangle ABC$ is inscribed in a circle and $\angle B > \angle C$, then the perpendicular from the center upon AB is greater than the perpendicular from the center upon AC.



Ex. 7

Ex. 9. The shortest chord through a point within a circle is the chord perpendicular to the radius drawn through the point.

Ex. 10. Two chords drawn from a point in a circle are unequal if they make unequal angles with the radius drawn to that point.

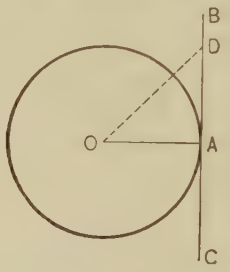
* Ex. 11. Two chords drawn through an interior point are unequal if they make unequal angles with the radius drawn through that point.

Ex. 12. *A diameter is greater than any noncentral chord.*

HINT. Draw radii to ends of chord. Their sum equals a diameter.

Proposition IX. Theorem

203. *The straight line perpendicular to a radius at its outer extremity is a tangent to the circle.*



Given in $\odot O$, radius $OA \perp BC$ at A.

To prove BC is a tangent.

Proof

STATEMENTS	REASONS
1. Join any other point D in BC to O.	1. Why?
2. $OD > OA$.	2. (131).
3. D lies without the circle.	3. (179).
4. $\therefore BC$ is a tangent to $\odot O$.	4. (175).

204. Cor. 1. *A tangent is perpendicular to the radius drawn to the point of contact.* (Converse to Prop. IX.)

Hint. The shortest line is the perpendicular, since the $\perp$ is the shortest line (identity).

205. Cor. 2. *The perpendicular to a tangent at the point of contact passes through the center of the circle.*

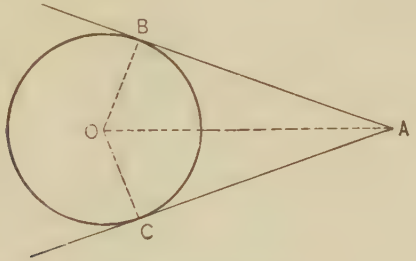
206. Cor. 3. *The perpendicular from the center to a tangent meets it at the point of contact.*

207. Cor. 4. *At a given point of contact there can be one tangent only.*

208. Def. The **length of a tangent** drawn from a point to a circle is the length of the line from this point to the point of contact, as AB (Prop. X).

Proposition X. Theorem

209. *The tangents drawn to a circle from a point without are equal.*



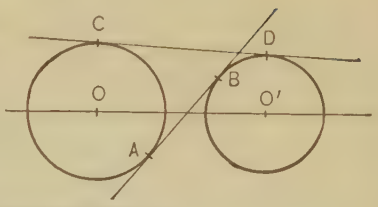
Given tangents AB and AC to circle O.

To prove $AB = AC$.

Proof

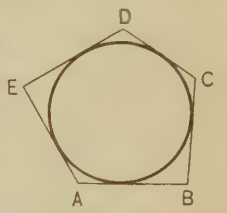
STATEMENTS	REASONS
1. Draw OA, OB, OC.	1. Why?
2. $\angle OBA$ and $\angle OCA$ are rt. $\measuredangle$.	2. Prop. IX, Cor. 1.
3. In rt. $\triangle OBA$ and OCA, $OB = OC$.	3. Why?
4. $OA = OA$.	4. Why?
5. $\triangle OBA \cong \triangle OCA$.	5. hy. a. = hy. a.
6. $\therefore AB = AC$.	6. Why?

210. Cor. *The tangents drawn to a circle from a point without make equal angles with the line joining the point to the center* (*i.e.* $\angle OAB = \angle OAC$).



211. DEF. A common tangent to two circles is called an **internal tangent** if the centers (O and O') are on opposite sides of the tangent, as AB. Otherwise it is called a **common external tangent**, as CD.

The **length** of a common tangent is the length of the segment between the **points of contact.**



212. DEF. A polygon is **circumscribed about a circle** if all its sides are tangent to the circle, as $ABCDE$. The **circle** is then said to be **inscribed** in the polygon.

The center of the inscribed circle is called the *incenter* of the polygon.

Ex. **1.** *The common internal tangents of two circles are equal.*

Ex. **2.** *The common external tangents of two circles are equal.*

Ex. **3.** A chord forms equal angles with the tangents drawn at its ends.

Ex. **4.** The sum of two opposite sides of a circumscribed quadrilateral is equal to the sum of the other two sides.

Ex. **5.** If hexagon $ABCDEF$ is circumscribed about a circle, $AB + CD + EF = BC + DE + FA$.

Ex. **6.** The sum of the arms of a right triangle circumscribed about a circle is equal to the hypotenuse increased by the diameter of the circle.

Ex. **7.** If two tangents make an angle of 60°, the chord joining the points of contact equals either tangent.

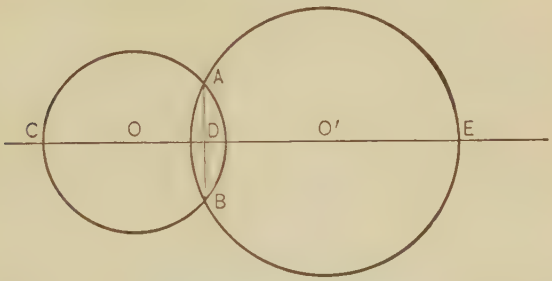
* Ex. **8.** Triangle ABC is circumscribed about a circle, which touches the sides AB, BC, and CA in X, Y, and Z, respectively. Find the lengths of AX, BY, and CZ if $AB = 3$, $BC = 4$, and $CA = 5$.

213. DEF. The **line of centers** is the line determined by the centers of two circles.

Thus OO' is the line of centers of circles O and O'. See figure at top of page.

PROPOSITION XI. THEOREM

214. *If two circles intersect, the line of centers bisects their common chord at right angles.*



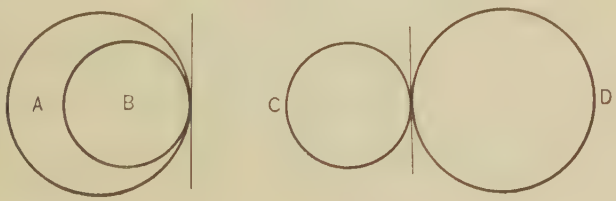
Given circles ACB and AEB intersecting at A and B.

To prove OO', their line of centers, is the perpendicular-bisector of AB.

Proof. O and O' are each equally distant from A and B. (Why?)

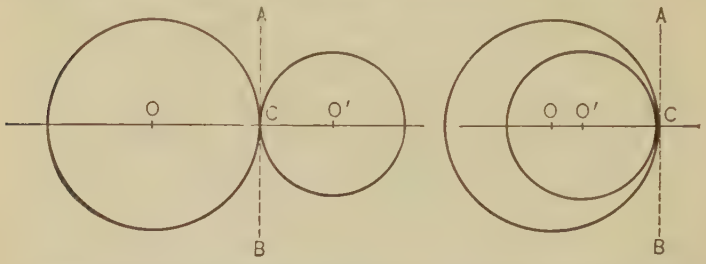
$\therefore$ OO' is the perpendicular-bisector of AB. (79)

215. DEF. Two circles are **tangent to each other** if both are tangent to the same straight line at the same point. They are tangent internally or externally, according as one circle lies within or without the other. Thus, circles A and B are tangent internally, and circles C and D are tangent externally.



Proposition XII. Theorem

216. *If two circles are tangent to each other, their line of centers passes through the point of contact.*



Given OO' is the line of centers of circles O and O' that are tangent at C.

To prove OO' passes through C.

Proof. Draw common tangent AB at C. At the point of contact, C, draw a perpendicular to the common tangent.

This perpendicular passes through O and O'. (205)

$\therefore$ OO' passes through C. (Why?)

Ex. 1. What are the relative positions of two circles, if the distance between their centers is

(*a*) Greater than the sum of the radii?
(*b*) Equal to the sum of the radii?
(*c*) Less than the sum but greater than the difference of the radii?
(*d*) Equal to the difference of the radii?
(*e*) Less than the difference of the radii?
(*f*) Equal to zero?

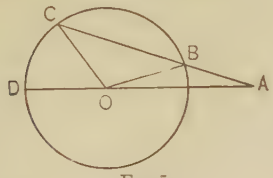
Ex. 2. If in the diagram of the preceding proposition (XII) tangents are drawn from A to circles O and O', these tangents are equal.

Ex. 3. If a secant intersects two concentric circles, its segments, intercepted by the two circles, are equal.

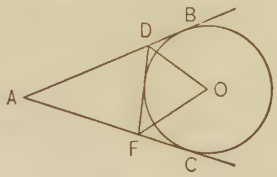
Ex. 4. Two parallel chords, drawn from the extremities of a diameter, are equal.

Ex. 5. If the radius OB is equal to AB, prove $\angle COD = 3\angle A$.

Ex. 6. If from A the tangents AB and AC are drawn to a given circle O, and a third tangent intersects AB and AC in D and F,



Ex. 5

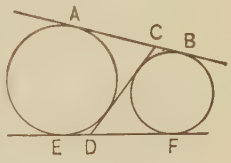


Ex. 6

Prove $AD + DF + AF = 2\,AB$,

and $\angle O = \text{rt. } \angle - \frac{\angle A}{2}$.

* Ex. 7. The two common external tangents of two circles intercept on a common internal tangent a segment, CD, equal to the external tangent, AB.



Ex. 7

MEASUREMENT

217. To **measure** a quantity is to find how many times it contains another quantity of the same kind, called a **unit of measure.**

218. Def. The **numerical measure** of a quantity is the number that expresses how many times the quantity contains the unit.

Thus a line is measured by finding the number of yards or inches that it contains. If the result is 16 inches, then 16 is the numerical measure of the line.

219. Def. A number is **rational** if it can be expressed as a common fraction, proper or improper.

Obviously this definition includes all integers.

Thus, $\frac{2}{7}$, 9, $2\frac{3}{4}$, 11.4 are rational numbers. Any rational number can be represented in the form $\frac{m}{n}$, in which m and n are integers.

220. Def. A number is **irrational** if it cannot be expressed by a common fraction, proper or improper.[1]

Thus $\sqrt{2}$ cannot be a fraction, for if $\sqrt{2}$ were equal to $\frac{m}{n}$ (where m and n are integers that have no common factor), then $2 = \frac{m \cdot m}{n \cdot n}$, a condition which is impossible since m and n have no common factors. Hence $\sqrt{2}$ is an irrational number. Other irrational numbers are $\frac{\sqrt{7}}{5}$, $\sqrt[3]{9}$, $\sqrt[5]{11}$, etc.

221. In the following chapters we shall designate as *successive approximate values* the decimals that we obtain if we calculate the values of irrational numbers to one, two, three, etc., decimal places.

Thus the successive approximate values of $\sqrt{2}$ are:

1.4, 1.41, 1.414, 1.4142, etc.

Since the true value of $\sqrt{2}$ lies between 1.4 and 1.5, the first approximation differs from the true value by less than .1. Similarly the second approximation differs from the true value by less than .01, the third by less than .001, etc. It is evident that by continuing the decimal this difference may be diminished to less than one millionth or one billionth or less than any assigned value, however small.

222. Irrational numbers cannot be expressed by a common fraction, proper or improper, but approximate values can be found that differ from the true value by less than any assigned value, however small.

223. Def. Two irrational numbers are called *equal* if *all* their successive approximate values are respectively equal.

Obviously two numbers that satisfy this condition cannot differ by any value, however small.

[1] The definition of "irrational" as given in many texts on elementary algebra refers to a special kind of irrational numbers only, viz. those that can be obtained by evolution. This is the only kind of irrational numbers that can be obtained by the operations of elementary algebra.

Thus, it can be shown that all successive approximate values of $\sqrt{3}$ and $\frac{3}{\sqrt{3}}$ are respectively equal, no matter to how many places of decimals we carry the calculation. Hence we call $\sqrt{3}$ and $\frac{3}{\sqrt{3}}$ equal irrational numbers.

224. DEF. The **ratio** of two quantities of the same kind is equal to the quotient of their numerical measures expressed in terms of the same unit.

Thus, the ratio of two quantities, a and b, is $\frac{a}{b}$ or $a \div b$; the ratio of four yards and two yards is $\frac{4}{2}$, or 2. A ratio is used to compare the magnitude of two quantities.

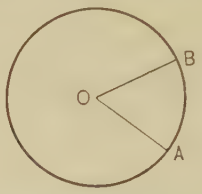
225. If the ratio of two quantities is a rational number, there always exists a third quantity, called a *common measure*, which is contained an integral number of times in each.

Thus, if AB and CD designate the length of two lines expressed in terms of the same unit, and $\frac{AB}{CD} = \frac{5}{9}$, then the ninth part of CD is contained in AB five times. Or if CD be divided into nine equal parts and one of these parts be laid off on AB, as often as possible, then AB contains 5 of these parts. Each of these parts is a common measure of AB and CD.

226. If the ratio of two quantities is an irrational number, it can be shown that they have no common measure; *e.g.* if $\frac{AB}{CD} = \sqrt{7}$, AB and CD cannot have a common rational measure, for otherwise $\sqrt{7}$ would equal a rational number.

Proposition XIII. Theorem[1]

227. *A central angle is measured by its intercepted arc.*



Given central $\angle O$ intercepting $\overset{\frown}{AB}$.

To prove $\angle O$ and $\overset{\frown}{AB}$ are measured by the same number of degrees.

Proof. Case I. The number of degrees in $\angle O$ is rational.

Then the number of degrees in $\angle O$ can be represented by $\frac{m}{n}$, where m and n are integers.

If we construct 360 equal angles about O, each angle would intercept $\frac{1}{360}$ of the circle. (183)

Or, A central $\angle$ of 1° intercepts an arc of 1°.

If the angle of 1° be divided into n equal parts, all intercepted arcs would be equal.

Or, A central $\angle$ of $\left(\frac{1}{n}\right)^\circ$ intercepts an arc of $\left(\frac{1}{n}\right)^\circ$.

Taking m of these parts, we have

A central $\angle$ of $\left(\frac{m}{n}\right)^\circ$ intercepts an arc of $\left(\frac{m}{n}\right)^\circ$.

Case II. The numbers of degrees in $\angle O$ is irrational.

Let the number of degrees in $\angle O$ be some irrational number, *e.g.* $\sqrt{2} = 1.41421\cdots$, and let us consider the successive approximate values of this number.

[1] Teachers who prefer the proof which is based upon the principle of limits will find the same in the appendix, pp. 275–277.

If $\angle O$ contained 1.4°, $\overset{\frown}{AB}$ contains 1.4°. (Case I.)

If $\angle O$ contained 1.41°, $\overset{\frown}{AB}$ contains 1.41°. (Case I.)

If $\angle O$ contained 1.414°, $\overset{\frown}{AB}$ contains 1.414°. (Case I.)

In other words all approximations of the numerical measures of $\angle O$ and of $\overset{\frown}{AB}$ are respectively equal.

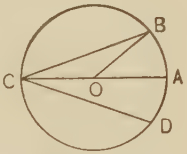
$\therefore$ the numerical measures of $\angle O$ and $\overset{\frown}{AB}$ are equal. (223)

NOTE. In the above demonstration a concrete numerical value is assumed for $\angle O$. It is evident, however, that the proof does not at all depend upon the particular value selected; in other words, that the conclusion follows in like manner for any other numerical value of $\angle O$. Hence the proof is general. For a more general form of proofs of this type see § 347.

228. COR. *In the same circle, or in equal circles, two central angles have the same ratio as their intercepted arcs.*

EX. 1. If $\overset{\frown}{AB} = 40°$, how many degrees are (*a*) in central $\angle AOB$? (*b*) in $\angle ACB$?

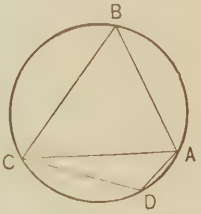
EX. 2. If $\overset{\frown}{BD} = 60°$, how many degrees are in $\angle C$?



EXS. 1, 2

229. DEF. An **inscribed angle** of a circle is an angle whose vertex lies in the circle and whose sides are chords; as $\angle B$, $\angle D$, $\angle BCD$, $\angle BCA$.

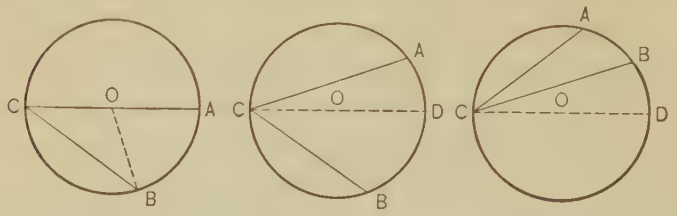
230. DEF. A **segment** of a circle is a figure formed by an arc and its chord, as arc *CBA* and chord *CA*.



231. DEF. An *angle* is said to be **inscribed in a segment** if its vertex lies in the arc and its sides pass through the extremities of that arc. Thus $\angle D$ is inscribed in segment *CDA*, and $\angle B$ in segment *CBA*.

Proposition XIV. Theorem

232. *An inscribed angle is measured by one half the intercepted arc.*



Given $\angle ACB$ inscribed in $\odot O$.

To prove $\angle ACB$ is measured by $\frac{1}{2}\overset{\frown}{AB}$.

Case I. One side of the angle is a diameter of the circle.

Proof

STATEMENTS	REASONS
1. Draw OB.	1. Why?
2. $BO = OC$.	2. Why?
3. $\angle C = \angle B$.	3. Base $\measuredangle$ of an isos. $\triangle$ are $=$.
4. $\angle AOB = \angle C + \angle B$.	4. (120).
5. $\angle AOB = 2\angle C$.	5. Sub.
6. $\angle ACB = \frac{1}{2}\angle AOB$.	6. Why?
7. $\angle AOB$ is measured by $\overset{\frown}{AB}$.	7. A central $\angle$ is meas. by its arc.
8. $\therefore \angle ACB$ is measured by $\frac{1}{2}\overset{\frown}{AB}$.	8. Why?

Case II. The center lies within the angle.

Draw the diameter CD.

Then $\angle ACD$ is measured by $\frac{1}{2}\overset{\frown}{AD}$. (Case I.)

and $\angle BCD$ is measured by $\frac{1}{2}\overset{\frown}{BD}$. (Case I.)

$\therefore \angle ACD + \angle BCD$ is measured by $\frac{1}{2}(\overset{\frown}{AD} + \overset{\frown}{BD})$. (Ax. 2.)

Or $\angle ACB$ is measured by $\frac{1}{2}\overset{\frown}{AB}$.

CASE III. The center lies without the angle.

Draw the diameter CD.

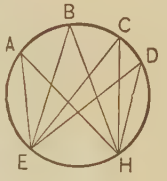
Then $\angle ACD$ is measured by $\frac{1}{2}\overset{\frown}{AD}$. (Case I.)

and $\angle BCD$ is measured by $\frac{1}{2}\overset{\frown}{BD}$. (Case I.)

$\therefore \angle ACB$ is measured by $\frac{1}{2}\overset{\frown}{AB}$. (Ax. 3.)

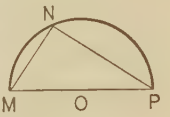
233. COR. 1. *Angles inscribed in the same segment are equal.*

$(\angle A = \angle B = \angle C = \angle D)$.



234. COR. 2. *An angle inscribed in a semicircle is a right angle* $(\angle N = 90°)$.

NOTE. The statement "$\angle AOB$ is measured by $\overset{\frown}{AB}$" is a brief form for writing an equation, viz. "the numerical measure of $\angle AOB$ = the numerical measure of $\overset{\frown}{AB}$." Hence, this and similar statements may be treated like equations, *e.g.* the axioms 1, 2, and 8 may be applied to them.



235. COR. 3. *The opposite angles of an inscribed quadrilateral are supplementary.*

Ex. 1. If in the diagram for Case I, $\angle C = 30°$, how many degrees are in $\overset{\frown}{CB}$?

Ex. 2. If in the same diagram $\overset{\frown}{BC} = 3\ \overset{\frown}{AB}$, find $\angle C$.

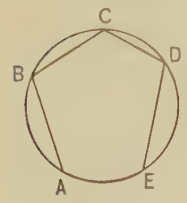
Ex. 3. If in the diagram for Case II $\overset{\frown}{AC}$ is $\frac{1}{3}$, and $\overset{\frown}{BC}$ is $\frac{1}{4}$ of the circle, find $\angle ACB$, $\angle ACD$.

Ex. 4. If in the diagram for Case III, A is the mid-point of $\overset{\frown}{CD}$, and B is the mid-point of $\overset{\frown}{AD}$, how many degrees are there in $\angle ACB$?

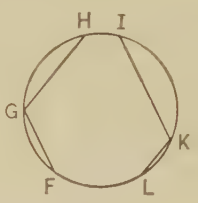
Ex. 5. If a quadrilateral $ABCD$ be inscribed in a circle, and the two diagonals be drawn, find all the angles in the figure, if $\overset{\frown}{AB} = 80°$, $\overset{\frown}{BC} = 110°$, and $\overset{\frown}{CD} = 90°$.

Ex. 6. In a diagram of the preceding exercise, find four pairs of equal angles.

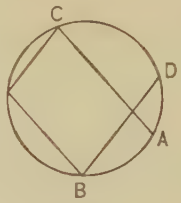
Ex. 7. Find $\angle B + \angle D$, if $\overset{\frown}{AE} = 40°$.



Ex. 7



Ex. 8



Ex. 9

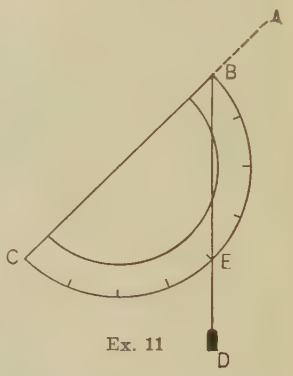
Ex. 8. How many degrees are in $\angle G + \angle K$, if $\overset{\frown}{HI} = 20°$, and $\overset{\frown}{FL} = 30°$?

Ex. 9. How many degrees are in $\angle B + \angle C$, if $\overset{\frown}{DA} = 50°$?

Ex. 10. If through one of the points of intersection of two equal circles a line be drawn to meet the circles, the extremities of that line are equidistant from the other point of intersection.

Ex. 11. If a semicircle is placed so that the prolongation of its diameter CB passes through a point A, and a plumb line BD intersects the semicircle at E, the angle of elevation of A is measured by $\frac{1}{2}\ \overset{\frown}{BE}$.

[For the definitions of angle of elevation and additional practical applications, see problems 34–36, p. 287.]



Ex. 11

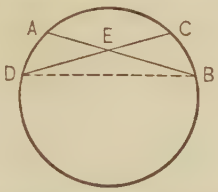
Ex. 12. A circle is divided into four parts in the continued ratio of 2 : 4 : 4 : 5. The points of division are joined in succession. Find the number of degrees in the inscribed angles thus formed.

Ex. 13. Lines are drawn on the face of a clock from 1 to 5 to 7 to 11 to 1. Find the number of degrees in each inscribed angle.

Ex. 14. An equilateral hexagon is inscribed in a circle. How many degrees are there in each angle of the hexagon?

Proposition XV. Theorem

236. *An angle formed by two chords intersecting within the circle is measured by one half the sum of its intercepted arc and the arc of its vertical angle.*



Given two chords AB and CD intersecting in E in $\odot O$.

To prove $\angle AED$ is measured by $\frac{1}{2}(\overset{\frown}{AD} + \overset{\frown}{CB})$.

Proof

STATEMENTS	REASONS
1. Draw DB.	1. Why?
2. $\angle AED = \angle D + \angle B$.	2. Ext. $\angle$ of $\triangle = \cdots$.
3. $\begin{cases} \angle D \text{ is meas. by } \frac{1}{2}\overset{\frown}{CB}. \\ \angle B \text{ is meas. by } \frac{1}{2}\overset{\frown}{AD}. \end{cases}$	3. Inscr. $\measuredangle$ $\cdots$.
4. $\angle D + \angle B$ is meas. by $\frac{1}{2}(\overset{\frown}{CB} + \overset{\frown}{AD})$.	4. Why?
5. $\therefore$ $\angle AED$ is meas. by $\frac{1}{2}(\overset{\frown}{CB} + \overset{\frown}{AD})$.	5. Why?

Ex. 1. If in diagram for Prop. XV, $\overset{\frown}{AD} = 60°$, $\overset{\frown}{BA} = 140°$, and $\overset{\frown}{CB} = 20°$, find $\angle AEC$ and $\angle DAE$.

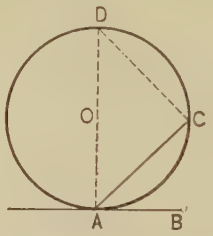
Ex. 2. If two perpendicular chords intersect within the circle, the sum of a pair of opposite intercepted arcs is equal to a semicircle.

Ex. 3. If in diagram for Prop. XV, $\overset{\frown}{AC}$ and $\overset{\frown}{DB}$ add to 210°, find $\angle CEB$.

Ex. 4. In same diagram (Prop. XV) $\angle B = 40°$, and $\overset{\frown}{BC} = 100°$. Find $\angle AED$.

Proposition XVI. Theorem

237. *An angle formed by a tangent to a circle and a chord drawn from the point of contact is measured by half the intercepted arc.*



Given AB a tangent, and AC a chord in $\odot O$.

To prove $\angle CAB$ is measured by $\frac{1}{2}\overset{\frown}{AC}$.

Proof

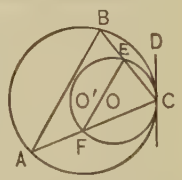
STATEMENTS	REASONS
1. Draw the diameter AD and chord DC.	1. Why?
2. $\angle C$ is a rt. $\angle$.	2. (234).
3. $\angle DAB$ is a rt. $\angle$.	3. (204).
4. $\angle D$ is compl. to $\angle DAC$.	4. (112).
5. $\angle BAC$ is compl. to $\angle DAC$.	5. Why?
6. $\angle BAC = \angle D$.	6. Why?
7. $\angle D$ is meas. by $\frac{1}{2}\overset{\frown}{AC}$.	7. Why?
8. $\therefore$ $\angle BAC$ is meas. by $\frac{1}{2}\overset{\frown}{AC}$.	8. Equal $\measuredangle$ have same measure.

Ex. **1**. If, in diagram for Prop. XVI, $\overset{\frown}{AC} = 2\,\overset{\frown}{CD}$, find $\angle CAB$.

Ex. **2**. A chord is parallel to the tangent drawn through the mid-point of the subtended arc.

Ex. **3**. Circles O and O' are tangent to DC at C. Prove that

$$\angle DCB = \angle A = \angle EFC.$$

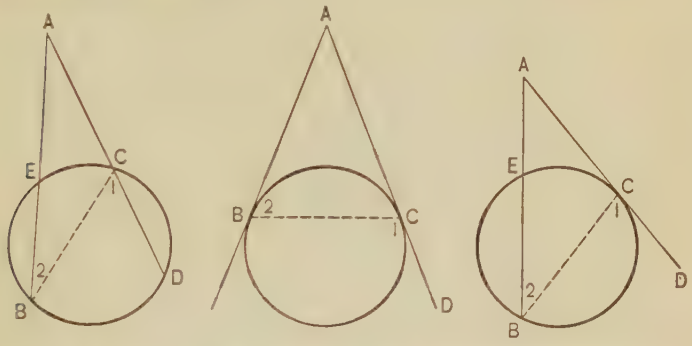


Ex. 3

Ex. **4**. Two tangents to a circle form an angle of 40°. How many degrees are there in the minor arc between the tangents?

Proposition XVII. Theorem

238. *An angle formed by two secants, or two tangents, or a tangent and a secant, intersecting without a circle, is measured by half the difference between the intercepted arcs.*



Case I. **Given** AB and AD secants drawn from an external point A to $\odot O$, cutting circle at E and C.

To prove $\angle A$ is measured by $\frac{1}{2}(\overset{\frown}{BD} - \overset{\frown}{EC})$.

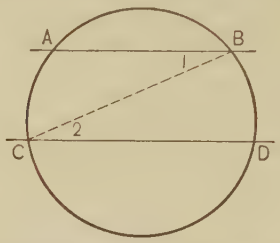
Proof

STATEMENTS	REASONS
1. Draw BC.	1. Why?
2. $\angle A + \angle 2 = \angle 1$.	2. Why?
3. $\angle A = \angle 1 - \angle 2$.	3. Why?
4. $\angle 1$ is meas. by $\frac{1}{2}\ \overset{\frown}{BD}$.	4. Ins. $\angle$ is meas. by···.
5. $\angle 2$ is meas. by $\frac{1}{2}\ \overset{\frown}{EC}$.	5. Same as 4.
6. $\angle 1 - \angle 2$ is measured by $\frac{1}{2}(\overset{\frown}{BD} - \overset{\frown}{EC})$.	6. Diff. of 2 $\measuredangle$ is meas. by diff. of their measures.
7. $\therefore$ $\angle A$ is measured by $\frac{1}{2}(\overset{\frown}{BD} - \overset{\frown}{EC})$.	7. Sub.

Note. Let the student prove Cases II and III.

Proposition XVIII. Theorem

239. *Parallel secants intercept equal arcs on a circle.*



Given AB and CD two parallel secants to $\odot O$.

To prove $\overset{\frown}{AC} = \overset{\frown}{BD}$.

Proof

STATEMENTS	REASONS
1. Draw CB.	1. Why?
2. $\angle 1$ is meas. by $\frac{1}{2}\ \overset{\frown}{AC}$. $\angle 2$ is meas. by $\frac{1}{2}\ \overset{\frown}{BD}$.	2. Ins. $\angle$ is meas. by $\frac{1}{2}$ its arc.
3. $AB \parallel CD$.	3. Hyp.
4. $\angle 1 = \angle 2$.	4. Alt. int. $\measuredangle$ of $\parallel$s are $=$.
5. $\frac{1}{2}\ \overset{\frown}{AC} = \frac{1}{2}\ \overset{\frown}{BD}$.	5. Equal $\measuredangle$ have $=$ measures.
6. $\therefore \overset{\frown}{AC} = \overset{\frown}{BD}$.	6. Why?

240. Cor. 1. *If a secant is parallel to a tangent, they intercept equal arcs on the circle.*

241. Cor. 2. *Two parallel tangents intercept equal arcs on the circle.*

Ex. 1. If the angle formed by two tangents is 60°, how many degrees are in each of the intercepted arcs?

Ex. 2. If an angle formed by a secant and a tangent is 20° and the greater of the intercepted arcs is 90°, how many degrees are in the other intercepted arc?

Ex. **3**. If in the diagram for Prop. XVII, Case I, $\overset{\frown}{BD} = 100°$ and $\angle A = 20°$, find $\overset{\frown}{EC}$.

Ex. **4**. If in the same diagram $\overset{\frown}{EC} = 60°$ and $\overset{\frown}{EB} = \overset{\frown}{BD} = \overset{\frown}{CD}$, find $\angle A$.

Ex. **5**. In the diagram for Prop. XVIII, find $\angle ABC$ if $\overset{\frown}{AB} = 80°$ and $\overset{\frown}{CD} = 120°$.

Ex. **6**. The line bisecting an angle formed by a tangent and a chord drawn to the point of contact, bisects the intercepted arc.

Ex. **7**. *An inscribed trapezoid is isosceles.*

Ex. **8**. In diagram for Prop. XVIII, if $\overset{\frown}{AB} = 80°$ and $\overset{\frown}{CD} = 200°$, find $\angle B$.

Ex. **9**. An exterior angle of an inscribed quadrilateral is equal to its opposite interior angle.

Ex. **10**. Find the sum of three alternate angles of an inscribed hexagon.

Ex. **11**. If in the greater of two concentric circles, chords be drawn touching the smaller circle, the chords are equal.

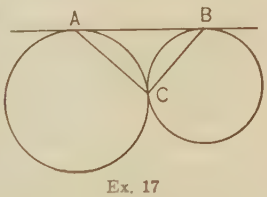
Ex. **12**. If two equal chords intersect, the lines joining their ends form an isosceles trapezoid.

Ex. **13**. If the bisector of an inscribed angle be produced until it meets the circle, and through this point of intersection a chord be drawn parallel to one side of the angle, it is equal to the other side.

Ex. **14**. A circle constructed on an arm of an isosceles triangle as diameter bisects the base.

Ex. **15**. If two circles intersect in A and B and AC and AD are diameters of the two circles, then a line joining C and D passes through B.

Ex. **16**. If two circles are tangent to each other and through the point of contact two lines are drawn, each terminating in the two circles, the chords joining the ends of these lines are parallel.



Ex. 17

Ex. **17**. If two circles are tangent at C and a common exterior tangent touches the circles in A and B, the angle ACB is a right angle.

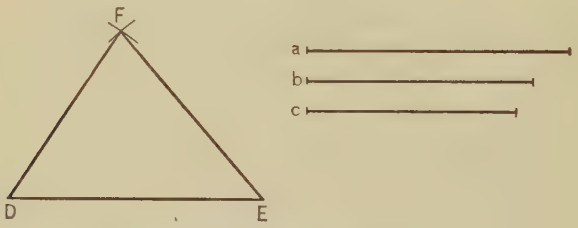
* Ex. **18**. If from the extremities of a diameter perpendiculars be drawn upon any chord (produced, if necessary), the feet of the perpendiculars are equidistant from the center.

CONSTRUCTIONS

242. Note. In the following examples, we shall denote the given parts of a triangle always in the same manner; the sides by a, b, c, the opposite angles by A, B, and C, the altitudes by h_a, h_b, and h_c, the medians by m_a, m_b, and m_c, and the angle-bisectors by t_a, t_b, and t_c.

Proposition XIX. Problem

243. *To construct a triangle, having given the three sides.* (*s. s. s.*)



Given a, b, and c the three sides of a $\triangle$.

Required to construct the triangle.

Construction. Draw $DE = a$.

From E as a center, with a radius equal to b, draw an arc.

From D as a center, with a radius equal to c, draw an arc.

The arcs intersect at F. Draw FE and FD.

$\triangle DEF$ is the required triangle.

Discussion. The construction is impossible if one side is greater than, or equal to, the sum of the other two.

Ex. 1. Construct an equilateral triangle, having one side given.

Ex. 2. *Construct an equilateral triangle whose perimeter equals a given line.*

Ex. 3. Construct a quadrilateral, having given the four sides and one diagonal.

Proposition XX. Problem

244. *To construct a triangle, having given two angles and the included side.* (*a. s. a.*)

The solution is left to the student.

Ex. 1. *Upon a given arm, to construct an isosceles right triangle.*

Ex. 2. Construct an isosceles triangle, having given the base and a base angle.

245. Remark. **The possibility of a solution of a problem depends often upon the proper choice of the part which is drawn first.**

Proposition XXI. Problem

246. *To construct a triangle, having given two sides and the included angle.* (*s. a. s.*)

The solution is left to the student.

Ex. 1. Construct an isosceles triangle, having given an arm and the vertex angle.

Ex. 2. *Construct a right triangle, having given the two arms.*

Ex. 3. To construct a quadrilateral ($ABCD$), having given three sides (AB, BC, and CD), the angle formed by the last of these sides and the unknown side ($\angle D$), and the angle formed by the same side and a diagonal ($\angle DCA$).

Ex. 4. To construct a pentagon ($ABCDE$), having given the five sides (AB, BC, CD, DE, and EA) and two angles including one of these sides ($\angle A$ and $\angle B$).

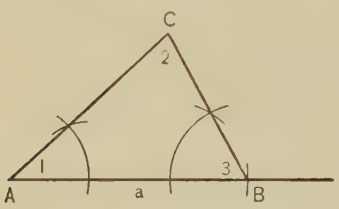
Ex. 5. Construct a quadrilateral, having given the four sides and one angle.

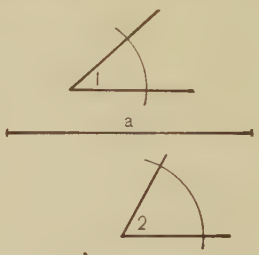
Is it possible to solve exercise five by constructing first a side not adjacent to the given angle?

Ex. 6. *To construct a rhombus having given the perimeter and one angle.*

Proposition XXII. Problem

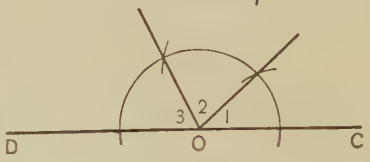
247. *To construct a triangle, having given one side, one adjacent and one opposite angle.* (*s. a. a.*)





Given $\angle 1$ and $\angle 2$ and side a opposite $\angle 2$.

Required to construct the $\triangle$.



Construction. In straight $\angle DOC$ at O lay off $\measuredangle$ 1, 2, as above. The remaining $\angle 3$ is equal to the third $\angle$ of the triangle.

The required $\triangle$ is ABC and is constructed by the method of Prop. XX. (a. s. a.)

Discussion. The construction is impossible if the sum of the given angles is greater than or equal to a straight angle.

Ex. **1.** Construct by means of Prop. XXII a right triangle, having given the hypotenuse and an acute angle.

Ex. **2.** Find a construction of the same problem which does not depend upon Prop. XXII.

Ex. **3.** *Construct a right triangle, having given an arm and the opposite angle.*

Ex. **4.** Construct an equilateral triangle, having given

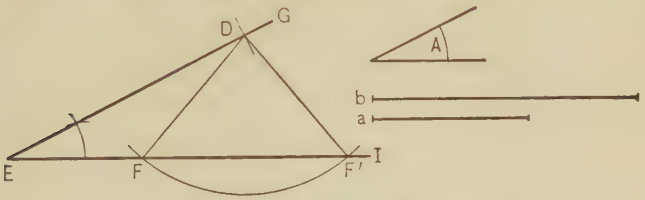
(*a*) *the altitude.*

(*b*) the radius of the inscribed circle.

(*c*) the radius of the circumscribed circle.

Proposition XXIII. Problem

248. *To construct a triangle, having given two sides and an angle opposite one of them.*



Given a and b, two sides of a $\triangle$, and $\angle A$ opposite side a.

Required to construct the $\triangle$.

Construction. Draw $\angle GEI = \angle A$.

On EG, lay off $ED = b$.

From D as a center with a radius equal to a, draw an arc intersecting EI in F and F'.

Both $\triangle EDF$ and EDF' fulfill the required conditions.

Discussion. If the arc intersects the base line twice on the same side of point E, there are two solutions, and if it touches the line or intersects once, there is one solution. If it does not touch the line, a solution is impossible.

Ex. 1. In Prop. XXIII, how many solutions are possible, when angle A is obtuse? Right? Acute?

Ex. 2. *Construct a right triangle, having given the hypotenuse and one arm.*

Ex. 3. Solve the preceding problem by drawing the hypotenuse first.

249. A triangle may be constructed if the following parts are given:

(1) Three sides.

(2) Two sides and the included angle.

(3) Two angles and the included side.

(4) Two angles and a non-included side.

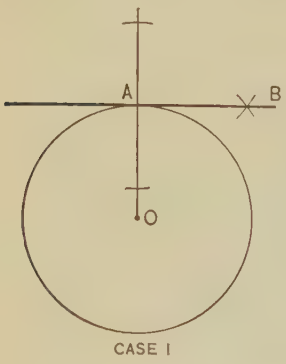
(5) Two sides and the angle opposite one of them.

To construct a triangle, three parts must be given, one part always being a side.

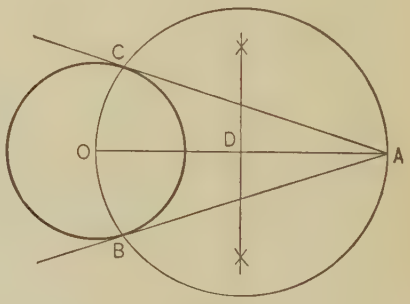
Ex. Are the three angles of a triangle three independent parts, and can a triangle be constructed when the three angles are given?

Proposition XXIV. Problem

250. *From a given point, to draw a tangent to a given circle.*



CASE I



CASE II

I. When the given point, A, is in the circle.

Hint. What is the angle formed by a radius and a tangent at its extremity?

II. When the given point, A, is without the circle.

Construction. Join A to O, the center of the given circle. From D, the mid-pt. of OA, as center and DA as a radius construct a circle intersecting the given circle in B and C.

Then AC and AB are the required tangents.

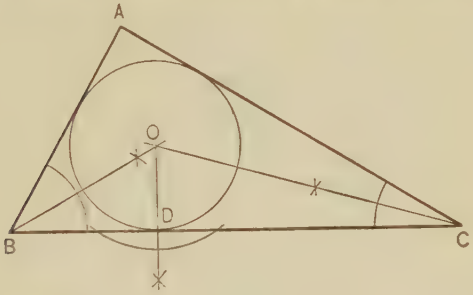
Hint. Show that $\measuredangle ACO$ and OBA are right angles.

Ex. 1. Construct a line tangent to a given circle and parallel to a given line.

Ex. 2. Construct a line tangent to a given circle and perpendicular to a given line.

Proposition XXV. Problem

251. *To inscribe a circle in a given triangle.*



Given $\triangle ABC$.

Required to inscribe a circle in $\triangle ABC$.

Construction. Construct the bisectors of $\angle B$ and $\angle C$.

These bisectors meet at some point O. (160)

From O draw $OD \perp BC$.

From O as a center, with a radius equal to OD, draw a circle, which is the required one.

[The proof is left to the student. See (160) and (203).]

252. Def. A circle touching one side of a triangle and the prolongations of the other two sides is an **escribed circle.**

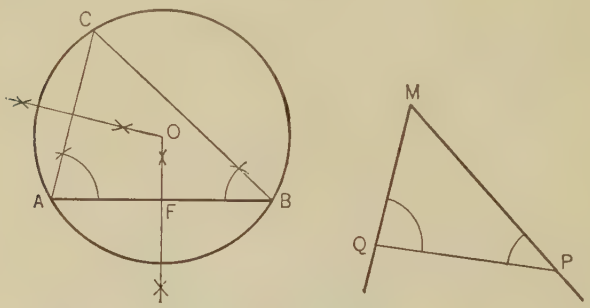
Ex. 1. Construct the three escribed circles of a triangle.

Ex. 2. The bisector of an angle of a triangle meets the circumscribed circle in a point which is equidistant from the other two vertices of the triangle and the center of the inscribed circle.

Ex. 3. A circle is inscribed in a triangle whose sides are 9, 8, and 9. Find the distances of the vertices from the points of contact (209).

Proposition XXVI. Problem

253. *Upon a given straight line as chord, to construct a segment of a circle which shall contain an angle equal to a given angle.*



Given line AB and $\angle M$.

Required to construct a segment of a circle on AB as chord which shall contain an angle equal to $\angle M$.

Construction. In the sides of $\angle M$ take any two points P and Q respectively.

Draw PQ.

On AB as a base construct $\triangle ABC$ so that $\angle A = \angle Q$, and $\angle B = \angle P$.

Circumscribe a circle about $\triangle ABC$.

Segment ACB is the required segment.

Proof. $\angle C = \angle M$. (113)

$\therefore$ Any angle inscribed in the segment $= \angle M$. (233)

Ex. On a line 2″ long construct a segment of a circle which shall contain an angle of 30°; an angle of 45°; an angle of 60°; an angle of 135°.

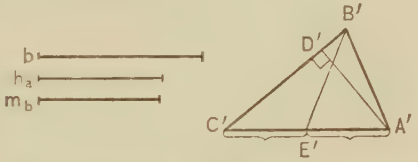
ANALYSIS OF PROBLEMS

254. An **analysis** of a problem is a course of reasoning by which its construction is discovered. Although no rules can be given which apply to all constructions, the method explained in the following exercises may be used in many problems.

Ex. A. *To construct a triangle having given one side, the corresponding median, and the altitude to another side.*

Given b, one side of triangle, m_b the corresponding median, and h_a the altitude upon another side.

Required to construct the $\triangle$.



Analysis. (1) Suppose $A'B'C'$ were the required triangle.

(2) Then we should know $C'A'(=b)$, $C'E'$ and $E'A' = \frac{b}{2}$, $A'D'(=h_a)$, $B'E'(=m_b)$, and $\measuredangle A'D'C'$ and $A'D'B'(=$ rt. $\measuredangle)$.

(The student is advised to mark the known parts or to draw them in a different color from the other lines.)

(3) Examine all triangles in the figure, and see if one can be constructed. The rt. $\triangle A'D'C'$ can be constructed, having given two sides.

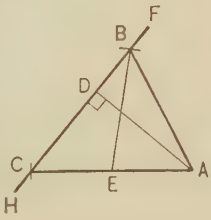
(4) Make this triangle the basis of the construction. Hence,

Construction. Draw $DA = h_a$.

At D, draw $FH \perp AD$.

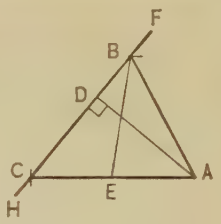
From A as a center, with a radius equal to b, draw an arc intersecting DH in C.

Draw AC.



Bisect AC in E, and from E as a center, with a radius equal to m_b, describe an arc, meeting FH in B.

ABC is the required triangle.



Proof. $AD = h_a$.

$BE = m_b$.

$AC = b$.

AD is an altitude, as

$\angle ADC$ = rt. $\angle$.

$CE = EA$

$\therefore$ EB is a median.

NOTE. The student should first carefully note whether the parts given are such as to make the construction possible. In Ex. A, the construction is impossible if $AD > AC$.

255. The following rules express the procedure in a general form:

1. Make a diagram resembling the one required, but not necessarily having the same dimensions.

2. Determine (*a*) all lines, (*b*) all angles, that are directly given, or that can be easily found from the given parts, and mark these parts.

3. Examine all triangles of the diagram until you discover a triangle that can be constructed.

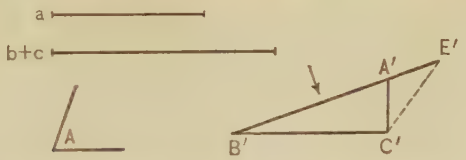
4. Make this triangle the basis of the construction, and try to determine successively all other parts of the figure.

5. In case no triangle can be found that can be constructed directly, draw additional lines which will enable you to obtain such a triangle.

256. Some problems require the drawing of additional lines (as stated above under 5 in section 255). In particular:

When a sum or a difference is given, construct such sum or difference in the analysis.

Ex. B. *To construct a triangle, having given the base, the sum of the other two sides, and the angle included by the two.*



Analysis. 1. Suppose $A'B'C'$ were the required triangle.

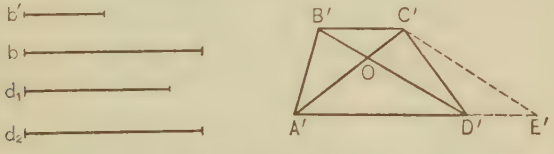
2. Then we should know two parts only.

$\therefore$ produce BA' to E' so that $A'E' = A'C'$.

Draw $E'C'$. We now know $C'B'$ $(=a)$, $B'E'(=b+c)$, $\angle B'A'C'(=\angle A)$, $\angle E'\left(=\frac{\angle A}{2}\right)$, and $\angle A'C'E'\left(=\frac{\angle A}{2}\right)$.

3. Examine all triangles. As $\triangle B'C'E'$ can be constructed, make this the basis of the construction.

Ex. C. *To construct a trapezoid, having given each base* (b and b') *and each diagonal* (d_1 and d_2).



Analysis. 1. Suppose $A'B'C'D'$ were the required trapezoid. Produce $A'D'$ to E' so that $D'E' = B'C'$.

Draw $C'E'$.

2. We know $A'E' = b + b'$, $A'C' = d_1$, $B'D' = C'E' = d_2$.

3. $\triangle A'C'E'$ can be constructed (s. s. s.).

4. $\square B'C'E'D'$ can now be constructed since we know two sides and the included angle.

5. A', B', C', D', the four vertices, are found so the trapezoid can be completed.

Construct a triangle, having given:

Ex. 1. a, b, m_b.

Ex. 2. a, b, h_b.

Ex. 3. $b, h_a, \angle A$.

Ex. 4. $b, t_c, \angle A$.

Ex. 5. $a, h_b, \angle B$.

Ex. 6. a, b, k_c (2 ▲).

Ex. 7. $\angle C, t_c, b$.

Ex. 8. $\angle A, \angle B, h_a$.

Ex. 9. a, h_a, m_a.

Ex. 10. b, h_a, m_a (2 ▲).

Ex. 11. a, m_a, m_b.

Ex. 12. a, h_b, h_c.

Ex. 13. $a, b, \angle A + \angle B$.

Ex. 14. $a, \angle B, b + c$.

Ex. 15. *Construct an isosceles triangle, having given the base and the vertex angle.*

Ex. 16. Construct an isosceles triangle, having given the sum of base and an arm, and a base angle.

Ex. 17. Construct a triangle, having given an angle, an adjacent side, and the difference of the other two sides.

Ex. 18. Construct a triangle, having given the base, the difference of the other two sides, and the angle included by the two sides.

Construct a right triangle, having given:

Ex. 19. *One arm and the altitude upon the hypotenuse.*

Ex. 20. The hypotenuse and the difference between the arms.

Ex. 21. The hypotenuse and the sum of the arms.

Ex. 22. *Construct a parallelogram, having given the sides and a diagonal.*

LOCI

257. A locus of a point in a plane is a line or a group of lines, such that (*a*) all points in the line (or lines) satisfy a certain condition, and (*b*) all points satisfying the condition are in the line (or lines).

Thus (*a*) Every point in the perpendicular-bisector of a line is equidistant from the ends of a line. (125)

(*b*) Every point equidistant from the ends of a line lies in the perpendicular-bisector of the line. (80)

∴ The perpendicular-bisector is the *locus* of the point that is equidistant from the ends of the line.

258. The word *locus* (plural *loci*) means "place," and it signifies in geometry the place of a point that moves according to certain conditions.

Thus, if one end of a stretched cord is fixed upon a drawing board, the other end, when moved about, will describe a circle. The circle is the locus of the other end.

Every point of an elevator car moving up or down moves in a straight line, which is perpendicular to the ground. Such a straight line is the locus of the moving point.

Similarly the locus of a point of the minute hand of a watch is a circle, etc.

259. To prove that a certain line is a locus, we must establish that:

(1) **Every point in the line or group of lines satisfies the given condition; and,**

(2) **Every point that satisfies the condition lies in the line or group of lines.** This is done by proving the theorem corresponding to (1) and by proving its converse corresponding to (2).

NOTE. We may establish (1) and (2′).

(2′) Every point outside the line or group of lines does *not* satisfy the given condition.

Thus, to find the locus of a point at a distance of $\frac{3}{8}$ inch from AB, it is not sufficient to prove that every point in CD has the proper distance from AB. It is also necessary to show that no point without satisfies the condition, which leads here to the discovery of another part of the locus, EF, located on the other side of AB.

EX. Draw the following loci without giving proofs:

(*a*) The locus of a point $\frac{3}{4}$ in. from a given point.

(*b*) The locus of a point $\frac{1}{4}$ in. from a given line.

(*c*) The locus of a point equidistant from a pair of parallel lines.

(*d*) The locus of the mid-point of all chords drawn in a given circle and parallel to a given line.

(*e*) The locus of the mid-points of all lines that are parallel to the base of a triangle and terminated by the other sides.

(*f*) The locus of a point equidistant from two given lines.

(*g*) The locus of the end of a line tangent to a given circle and $\frac{1}{4}$ in. long.

(*h*) The locus of the center of a circle whose radius equals $\frac{1}{4}$ in., and that touches a given circle (radius 1 in.) externally.

LOCUS THEOREMS

260. The locus of a point at a given distance from a given point is the circle described from the point as center with the given distance as radius.

For (*a*) every point in the circle has the required distance, while (*b*) every point at the required distance is in the circle.

261. The locus of a point equidistant from the ends of a given line is the perpendicular-bisector of that line.

For (*a*) every point in the perpendicular-bisector is equidistant. (*b*) Every point equidistant from the ends of the line lies in the perpendicular-bisector. (80)

262. The locus of a point that is at a given distance from a given straight line consists of two lines parallel to the given line at the given distance.

263. The locus of a point equidistant from two given parallel lines is a third parallel lying midway between the two given parallel lines.

264. The locus of a point equidistant from two intersecting straight lines consists of the bisectors of the included angles.

Ex. 1. Find the locus of the mid-points of the radii of a given circle.

Ex. 2. *Find the locus of the vertex of all right angles whose sides pass through two given points.*

Ex. 3. *Find the locus of the center of a circle which has a given radius and touches a given line.*

Ex. 4. Find the locus of the center of a circle which has a given radius and passes through a given point.

Ex. 5. *Find the locus of the center of a circle that passes through two given points.*

Ex. 6. *Find the locus of the center of a circle that touches two given lines.*

Ex. 7. Find the locus of the center of a circle which has a given radius and touches a given circle.

Ex. 8. *Find the locus of the center of a circle touching a given line at a given point.*

Ex. 9. *Find the locus of the center of a circle that touches a given circle in a given point.*

Ex. 10. Two vertices, B and C, of a triangle have a fixed position. Find the locus of the third vertex (A) if h_a equals a given line.

Ex. 11. Two vertices, B and C, of a triangle have a fixed position. Find the locus of the third vertex (A) if m_a equals a given line.

Ex. 12. The base of a parallelogram has a fixed position and length, and the adjacent side has a given length. Find the locus of the intersection of the diagonals.

Ex. 13. The base of a rectangle has a fixed position. Find the locus of the intersection of the diagonals.

Ex. 14. *Find the locus of the mid-points of all chords that have a given length and are drawn in a given circle.*

Ex. 15. Find the locus of the mid-points of all chords that can be drawn from a given point in the circle.

Ex. 16. The locus of the vertex of a triangle which has a given (fixed) base and a given vertex angle is an arc (constructed according to 253, p. 140).

265. Loci are used to determine the position of a point (or points) that satisfies two conditions. Each condition determines a locus, and the point (or points) of intersection of the two loci is the required point.

In the following exercises, state under what conditions no point, one point, or several points may be found.

Ex. 1. In a given line, AB, find a point at a given distance, d, from a given point C.

Ex. 2. In a given line, AB, find a point at a given distance, d, from a given line CD.

Ex. 3. In a given line, AB, find a point equidistant from two given points, P and Q.

Ex. 4. In a given circle, find a point at a given distance, d, from a given point C.

Ex. 5. In a given circle, find a point equidistant from two given parallel lines, CD and EF.

Ex. 6. In a given circle, find a point equidistant from two given intersecting lines, CD and EF.

Ex. 7. Find a point equidistant from two given intersecting lines, AB and CD, and at a given distance from a given point, E.

Ex. 8. Find a point equidistant from two given intersecting lines, AB and CD, and at a given distance from a given line, EF.

Ex. 9. Find a point equidistant from two given intersecting lines, AB and CD, and equidistant from two given points, E and F.

Ex. 10. Find a point equidistant from two given points, and having a given distance from a given point, E.

Ex. 11. Find a point equidistant from two given points and equidistant from two given parallel lines, EF and GH.

Ex. 12. Find a point equidistant from two given parallel lines and equidistant from two given intersecting lines, EF and GH.

Ex. 13. Find a point at a given distance d, from a given line, AB, and equidistant from two given points, E and F.

Ex. 14. Find a point having a given distance, d, from a given line, AB, and equidistant from two given parallel lines, EF and GH.

To construct a circle having a given radius:

Ex. 15. Touching a given line and passing through a given point.

Ex. 16. Touching two given lines.

Ex. 17. Passing through a point and touching a given circle.

Ex. 18. Touching two given circles.

Ex. 19. Touching a given circle and a given line.

To construct a circle:

Ex. 20. Touching a given line in a given point and passing through another point without.

Ex. 21. Touching a given circle at a given point and passing through another given point without.

Ex. 22. Touching a given line and a given circle at a given point.

HINT. Draw a tangent to the circle at the given point.

* Ex. 23. Construct a triangle having given a, h_a, $\angle A$.

Ex. 24. Construct a triangle having given a, m_a, $\angle A$.

266. No general method can be given for the solution of exercises; a great many, however, can be solved:

(1) **By a gradual putting together of the given parts.** (245)

(2) **By means of an analysis.**

(3) **By means of loci.**

MISCELLANEOUS EXERCISES

Construct an isosceles triangle, having given:

Ex. 1. The base and the altitude upon an arm.

Ex. 2. The altitude upon the base and the vertex angle.

Ex. 3. The vertex angle and the sum of one arm and the base.

Ex. 4. The perimeter and the base angles.

Construct a right triangle, having given:

Ex. 5. One acute angle and the altitude upon the hypotenuse.

Ex. 6. The altitude upon the hypotenuse and one of the segments of the hypotenuse.

Ex. 7. The sum of the arms and one acute angle.

Ex. 8. To find a point in one side of a triangle which is equidistant from the other two sides.

Ex. 9. *Find the locus of the vertex of a right triangle, having a given hypotenuse.*

Ex. 10. In one side of a quadrilateral to find a point equidistant from the ends of the opposite side.

Ex. 11. From a point P, in a circle, to draw a chord, having a given distance from the center.

Ex. 12. In a given circle, to draw a diameter having a given distance from a given point.

Ex. 13. Through a given point to draw a line, having a given distance from another point.

Ex. 14. Through two given points in a circle, to draw two equal parallel chords.

Ex. 15. *Trisect a given straight angle.*

Ex. 16. *Trisect a given right angle.*

Ex. 17. *Trisect a semicircle.*

Ex. 18. Through a given point outside a given line, to draw a line making a given angle with a given line.

Ex. 19. Through a given point, to draw a line of given length terminating in two given parallel lines.

Ex. 20. Through a given point, to draw a line making equal angles with two given lines.

* Ex. 21. To bisect an angle formed by two lines, without producing them to their intersection.

To construct a triangle, having given: (Note 242)

Ex. 22. a, $\angle B$, h_a.

Ex. 23. a, $\angle B$, m_c.

Ex. 24. a, $\angle B$, t_c.

Ex. 25. $\angle A$, h_a, t_a.

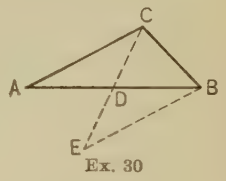
Ex. 26. $a + b + c$, $\angle B$, $\angle C$.

Ex. 27. a, b, R.[1]

Ex. 28. a, h_a, R.[1]

Ex. 29. h_a, h_b, $\angle B$.

Ex. 30. a, b, m_c.



Ex. 30

HINT. Produce m_c its own length.

* Ex. 31. a, m_c, $\angle C$.

* Ex. 32. m_a, m_b, h_c.

* Ex. 33. m_a, h_b, h_c.

* Ex. 34. m_a, m_b, m_c.

To construct a square, having given:

Ex. 35. *The diagonal.*

Ex. 36. The difference between the diagonal and the side.

Ex. 37. The sum of the diagonal and the side.

To construct a rectangle, having given:

Ex. 38. *One side and a diagonal.*

Ex. 39. *One side and the angle formed by the diagonals.*

Ex. 40. The perimeter and the diagonal.

[1] R = radius of circumscribed circle.

To construct a rhombus, having given:

Ex. **41**. *The two diagonals.*

Ex. **42**. The perimeter and one diagonal.

Ex. **43**. *One angle and a diagonal.*

Ex. **44**. *The altitude and the base.*

Ex. **45**. The altitude and one angle.

To construct a parallelogram, having given:

Ex. **46**. *Two adjacent sides and one altitude.*

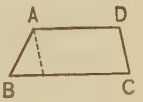
Ex. **47**. *Two adjacent sides and an angle.*

Ex. **48**. *One side and two diagonals.*

Ex. **49**. One side, one angle, and one diagonal.

Ex. **50**. *The diagonals and the angle formed by the diagonals.*

267. In the analysis of a problem relating to a trapezoid, draw a line through one vertex, A, either parallel to the opposite arm, DC, or parallel to a diagonal, DB.



To construct a trapezoid, having given:

Ex. **1**. *The four sides.*

Ex. **2**. *The bases and the lower base angles.*

Ex. **3**. The bases, another side, and one base angle.

Ex. **4**. The bases and the diagonals.

Ex. **5**. One base, the diagonals, and the angle formed by the diagonals.

Ex. **6**. *To draw a common external tangent to two given circles.*

Ex. **7**. *To draw a common internal tangent to two given circles.*

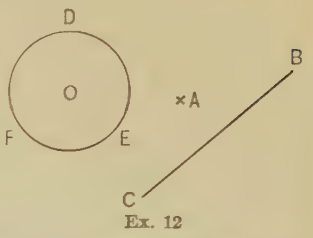
Ex. **8**. About a given circle, to circumscribe a triangle, having given the angles.

Ex. **9**. Find the locus of the mid-points of the secants that pass through a given point without a circle.

Ex. **10**. In a given circle, to inscribe a triangle, having given the angles.

* Ex. **11**. From a given point in a circle, to draw a chord that is bisected by a given chord.

Ex. **12**. Given a point, A, between a circle and a straight line. Through A, to draw a line terminated by the circle and the given line, and bisected in A.



Ex. 12

Ex. **13**. Given two points, A and B, on the same side of a line, CD. To find a point, X, in CD, such that $\angle AXC = \angle BXD$.

Ex. **14**. If two circles are circumscribed about two congruent triangles, the circles are equal.

Ex. **15**. If two circles are inscribed in two congruent triangles, the circles are equal.

Ex. **16**. To construct an isosceles triangle, given the base and the radius of the inscribed circle.

Ex. **17**. A circle is tangent internally at P to another circle of twice the radius. Prove that the small circle bisects every chord of the larger circle drawn from P.

Ex. **18**. Four points, A, B, C, D, on a circle divide it into parts proportional to 6, 9, 10, 11. Find the number of degrees in all angles formed by joining the four points in succession, drawing the diagonals, and extending pairs of sides until they meet outside the circle.

BOOK III

PROPORTION. SIMILAR POLYGONS

268. DEF. A **proportion** is a statement of the equality of two ratios, as $\frac{a}{b} = \frac{c}{d}$ or $a : b = c : d$ (see p. 123).

NOTE. The statements $\frac{a}{b} = \frac{c}{d}$, and $a : b = c : d$ are equivalent. Hence, if a hypothesis states $a : b = c : d$, we may in the proof employ this statement in the form $\frac{a}{b} = \frac{c}{d}$ without assigning a special reason. Similarly, if we have to prove $a : b = c : d$, we may prove instead $\frac{a}{b} = \frac{c}{d}$.

269. DEF. The first and the fourth terms of a proportion are called the **extremes**; the second and the third, the **means**.

270. DEF. The first and the third terms are called the **antecedents**, the second and the fourth the **consequents**.

Thus, in the proportion, $a : b = c : d$, a and d are the extremes, b and c the means, a and c the antecedents, and b and d the consequents.

271. DEF. When the means of a proportion are equal, *either* mean is said to be the **mean proportional** between the first and the last terms.

Thus, in the proportion, $a : b = b : c$, b is the mean proportional between a and c.

272. DEF. The last term is the **fourth proportional** to the first three.

Thus, in the proportion, $a : b = c : d$, d is the fourth proportional to a, b, and c.

273. The two terms of a ratio must be either quantities of the same denomination, or the quantities must be represented by their numerical measures only.

Proposition I. Theorem

274. *In any proportion, the product of the means is equal to the product of the extremes.*

Given $a : b = c : d.$

To prove $ad = bc.$

Proof. $\frac{a}{b} = \frac{c}{d}.$ (Hyp.)

Clearing of fractions, *i.e.* multiplying both members by bd,

$ad = bc.$ (Ax. 7.)

275. Cor. *If any three terms of a proportion are respectively equal to the three corresponding terms of another proportion, the remaining terms are equal.*

276. Note. The product of two quantities, in geometry, means the product of the numerical measures of the quantities.

Ex. **1.** Determine the value of x, if

(*a*) $3 : x = 4 : 8,$ (*c*) $x : 7 = 2 : 21,$

(*b*) $112 : 42 = 16 : x,$ (*d*) $a : m = x : n.$

Ex. **2.** Find the fourth proportional to

(*a*) 1, 2, and 3, (*b*) 2, 1, and 3, (*c*) m, n, and p.

Ex. **3.** Find the mean proportional to

(*a*) 9 and 4, (*b*) 30 and 3, (*c*) 1 and 16.

Proposition II. Theorem

277. *If the product of two numbers is equal to the product of two other numbers, either pair of numbers may be made the means, and the other pair the extremes, of a proportion.*

Given $ad = bc.$

To prove $a : b = c : d.$

Proof. $ad = bc.$ (Hyp.)

Dividing both members by bd,

$$\frac{a}{b} = \frac{c}{d}.$$ (Ax. 8.)

Ex. **1**. If $ab = mn$, find all possible proportions consisting of a, b, m, and n.

Ex. **2**. Form two proportions commencing with 3 from the equality $3 \times 10 = 5 \times 6$.

Ex. **3**. If $ab = xy$, form two proportions commencing with b.

Ex. **4**. Find the ratio of $x : y$, if

(*a*) $6x = 5y$,
(*b*) $9x = 2y$,
(*c*) $6x = y$,
(*d*) $mx = ny$,
(*e*) $(a + b)x = cy$,
(*f*) $mx + nx = py$,
(*g*) $4x - 6y = 0$,
(*h*) $ax - ay = bx - by$.

Proposition III. Theorem

278. *A mean proportional between two quantities is equal to the square root of their product.*

Given $a : b = b : c.$

To prove $b = \sqrt{ac}.$

Proof. $a : b = b : c.$ (Hyp.)

$b^2 = ac.$ (274)

Extracting the square root of both members,

$$b = \sqrt{ac}.$$

Ex. Find the mean proportional between

(*a*) 2 and 18.
(*b*) $\frac{4}{3}$ and $\frac{25}{3}$.
(*c*) $2a$ and $32a$.
(*d*) $3b^2$ and $9a^2$.

PROPOSITION IV. THEOREM

279. *If four quantities are in proportion, they are in proportion by alternation,* i.e. *the first term is to the third as the second is to the fourth.*

Given	$a : b = c : d.$	
To prove	$a : c = b : d.$	
Proof.	$\frac{a}{b} = \frac{c}{d}.$	(Hyp.)
	$ad = bc.$	(274)
	$\therefore a : c = b : d.$	(277)

PROPOSITION V. THEOREM

280. *If four quantities are in proportion, they are in proportion by inversion,* i.e. *the second term is to the first as the fourth is to the third*

Given	$a : b = c : d.$	
To prove	$b : a = d : c.$	
Proof.	$a : b = c : d.$	(Hyp.)
	$ad = bc.$	(274)
	$\therefore b : a = d : c.$	(277)

Ex. Transform the proportion, $m : x = p : q$, so that x becomes the fourth term; the first term; the third term.

PROPOSITION VI. THEOREM

281. *If four quantities are in proportion, they are in proportion by composition,* i.e. *the sum of the first two terms is to the second term as the sum of the last two terms is to the fourth term.*

Given	$a : b = c : d.$
To prove	$a + b : b = c + d : d.$

Proof. $a : b = c : d,$ (Hyp.)

or $\frac{a}{b} = \frac{c}{d}.$

$\frac{a}{b} + 1 = \frac{c}{d} + 1.$ (Why?)

$\frac{a}{b} + \frac{b}{b} = \frac{c}{d} + \frac{d}{d}.$ (Why?)

$\therefore \frac{a+b}{b} = \frac{c+d}{d},$ (Why?)

or $a + b : b = c + d : d.$

Proposition VII. Theorem

282. *If four quantities are in proportion, they are in proportion by division,* i.e. *the difference between the first two terms is to the second term as the difference between the last two terms is to the fourth term.*

Given $a : b = c : d.$

To prove $a - b : b = c - d : d.$

Proof. $\frac{a}{b} = \frac{c}{d}.$ (Hyp.)

$\frac{a}{b} - 1 = \frac{c}{d} - 1,$ (Ax. 3.)

$\therefore \frac{a-b}{b} = \frac{c-d}{d}.$ (Sub.)

Ex. **1.** Transform the following proportions so that only one term contains x.

(*a*) $2 : 3 = 5 - x : x.$ (*d*) $4 : 3 = 2 + x : x.$

(*b*) $6 : 7 = 2 - x : x.$ (*e*) $7 : 5 = 3 + x : x.$

(*c*) $a : b = 5 - x : x.$ (*f*) $a : b = 5 + x : x.$

Ex. **2.** If $x + y : y = 7 : 3$, find the ratio of x and y.

Ex. **3.** If $x - y : y = 2 : 3$, find the ratio of x and y.

Proposition VIII. Theorem

283. *If four quantities are in proportion, they are in proportion by composition and division,* i.e. *the sum of the first two terms is to their difference as the sum of the last two terms is to their difference.*

Given $a : b = c : d.$

To prove $a + b : a - b = c + d : c - d.$

Proof.

$a : b = c : d.$ (Hyp.)

$\frac{a+b}{b} = \frac{c+d}{d}.$ (281)

$\frac{a-b}{b} = \frac{c-d}{d}.$ (282)

$\therefore \frac{a+b}{a-b} = \frac{c+d}{c-d}.$ (Ax. 8.)

Ex. **1.** Transform the following proportions so that only one term contains x.

(*a*) $3 : 2 = 5 + x : 5 - x.$
(*b*) $5 : 3 = 3 + x : 3 - x.$
(*c*) $a : b = 1 + x : 1 - x.$

Ex. **2.** If $x + y : x - y = 12 : 5$, find the ratio of x to y.

Ex. **3.** If $x + y : x - y = a : b$, find the ratio of x to y.

Proposition IX. Theorem

284. *In a series of equal ratios, the sum of any number of antecedents is to the sum of the corresponding consequents as any antecedent is to its consequent.*

Given $a : b = c : d = e : f.$

To prove $a + c + e : b + d + f = a : b.$

Proof. Let $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} \cdots = r.$ (Hyp.)

Then $a = br,\ c = dr,\ e = fr,$ (Why?)

and $a + c + e = br + dr + fr$ (Ax. 2.)

or $a + c + e = (b + d + f)r$ (Sub.)

$\therefore \frac{a + c + e}{b + d + f} = r = \frac{a}{b} = \frac{c}{d} = \frac{e}{f}$ (Why?)

Ex. 1. If $a : b = c : d = e : f = 5 : 7$, find $\frac{a + c + e}{b + d + f}$.

Ex. 2. If $\frac{x - a - c}{y - b - d} = \frac{a}{b} = \frac{c}{d}$, find $x : y$.

Ex. 3. If $\frac{x}{2} = \frac{y}{3} = \frac{z}{4}$, find the ratio $x + y + z$ to z.

Proposition X. Theorem

285. *The products of the corresponding terms of two or more proportions are in proportion.*

Given $a : b = c : d.$

$m : n = p : q.$

To prove $am : bn = cp : dq$

Proof. $\frac{a}{b} = \frac{c}{d},\ \frac{m}{n} = \frac{p}{q}.$ (Hyp.)

$\therefore \frac{am}{bn} = \frac{cp}{dq}.$ (Ax. 7.)

286. Cor. If $\frac{a}{b} = \frac{c}{d},$

then $\frac{ma}{nb} = \frac{mc}{nd}.$

Ex. 1. If $x : y = 1 : 4$, and $x : \frac{1}{y} = 1 : 9$, find x.

Ex. 2. If $\frac{1}{x} : y = 1 : 2$, and $x : y = 1 : 8$, find y.

PROPOSITION XI. THEOREM

287. *If four quantities are in proportion, like powers or like roots of these quantities are in proportion.*

Given $a : b = c : d.$

To prove $a^n : b^n = c^n : d^n,$

and $\sqrt[m]{a} : \sqrt[m]{b} = \sqrt[m]{c} : \sqrt[m]{d}.$

Proof. $\frac{a}{b} = \frac{c}{d}.$ (Hyp.)

Raising both members to the nth power,

$$\therefore \frac{a^n}{b^n} = \frac{c^n}{d^n}.$$ (Why?)

Similarly, taking the mth root of each member,

$$\therefore \frac{\sqrt[m]{a}}{\sqrt[m]{b}} = \frac{\sqrt[m]{c}}{\sqrt[m]{d}}.$$ (Why?)

288. In elementary plane and solid geometry we need only squares and square roots, and cubes and cube roots. Prop. XI might read, for us, "If four quantities are in proportion, then the squares and square roots (also cubes and cube roots) of the quantities are in proportion."

Ex. **1.** If $x^3 : y^3 = 64 : 125$, find $\frac{x}{y}$.

Ex. **2.** If $\sqrt{x} : \sqrt{y} = 1 : 2$, find $x : y$.

Ex. **3.** If $\sqrt[3]{x} : \sqrt[3]{y} = 1 : 3$, find $x : y$.

Ex. **4.** If $\sqrt[3]{x} : 1 = \sqrt[3]{y} : 2$, find $\frac{x}{y}$.

Ex. **5.** If $x + y : x - y = 3 : 1$, find the ratio of x^2 to y^2.

Ex. **6.** If $x^2 : 4\,a^2 = y^2 : b^2$, find the ratio of x^3 to y^3.

Proposition XII. Theorem

289. *Equimultiples of two quantities are in the same ratio as the quantities.*

Given a and b two quantities.

To prove $ma : mb = a : b.$

Proof.

$a : b = a : b.$ (Iden.)

$m : m = 1 : 1.$ (Why?)

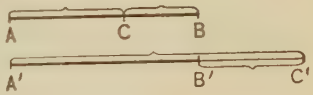
$\therefore ma : mb = a : b.$ (Prop. X.)

290. Def. If in a line AB, or its prolongation, a point C be taken, AC and BC are called **segments of the line.**

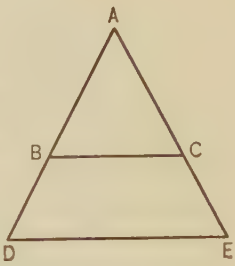
291. The segments are **internal** or **external** ones, according as C lies in AB or in the prolongation of AB.

Thus AB is divided internally, $A'B'$ externally.

The segments of AB are AC and BC. The segments of $A'B'$ are $A'C'$ and $B'C'$.



Note. If BC meets AD and AE so that $AB : BD = AC : CE$, the line BC is said to divide the sides into *segments that are proportional.* If $AB : AD = AC : AE$, then the sides are said to be *divided proportionally.*



Ex. 1. Divide a line 10 inches long internally in the ratio of 2 : 3.

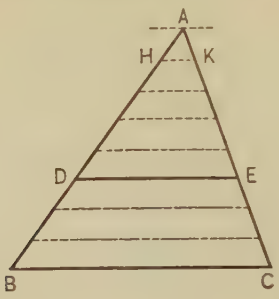
Ex. 2. Divide a line 18 inches long externally in the ratio of 3 : 2.

Ex. 3. An 18-inch line is divided into segments in the continued ratio of 2 : 3 : 4. Find the segments.

PROPORTIONAL LINES

Proposition XIII. Theorem[1]

292. *A line parallel to one side of a triangle divides the other two sides into segments that are proportional.*



Given in $\triangle ABC$, DE parallel to BC.

To prove $AD : DB = AE : EC$.

Proof

Case I. (Commensurable Case.)

STATEMENTS	REASONS
1. Let AH be the unit of measure so that $AD = m(AH)$ and $DB = n(AH)$.	1. By hyp., the two segments are commensurable.
2. $\frac{AD}{DB} = \frac{m}{n}$.	2. The ratio of two quantities of the same kind equals the ratio of their numerical measures.
3. Through point A and the various points of division, draw lines $\parallel$ to BC.	3. Why?
4. These lines are $\parallel$ to BC, DE, and to each other.	4. Why?

[1] For the proof based on principle of limits, see Appendix.

5. AE and EC will be divided into m and n segments, each equal to AK; *i.e.* $AE = m(AK)$ and $EC = n\ (AK)$.	5. ‖s cutting off equal distances on one transversal cut off equal distances on every transversal.
6. $\frac{AE}{EC} = \frac{m}{n}$.	6. Why?
7. $\therefore AD : DB = AE : EC$.	7. Why?

[Practical applications, pages 291–296]

CASE II. $\frac{AD}{DB}$ is an irrational number.

Let $\frac{AD}{DB} = \sqrt{3} = 1.732 \cdots$

If $\frac{AD}{DB} = 1.7$, then $\frac{AE}{EC} = 1.7$. (Case I.)

If $\frac{AD}{DB} = 1.73$, then $\frac{AE}{EC} = 1.73$, and so forth. (Case I.)

Hence *all* approximate values of $\frac{AD}{DB}$ and $\frac{AE}{EC}$ are respectively equal.

$$\therefore \frac{AD}{DB} = \frac{AE}{EC}. \tag{223}$$

293. REMARK. If the transversal intersects two sides of the triangle, these sides are divided internally in the same ratio; if it meets their prolongations, the sides are divided externally in the same ratio.

294. COR. I. *If a line parallel to one side of a triangle intersects the other two sides, either side is to one of its segments as the other side is to its corresponding segment.*

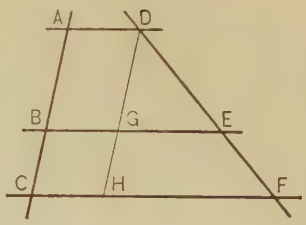
Proof: Using figure for the proposition, we can show that $AB = (m+n)(AH)$ and $AD = m(AH)$.

Hence, $\frac{AB}{AD} = \frac{m+n}{m}$.

Likewise, $\frac{AC}{AE} = \frac{m+n}{m}$ and $\therefore \frac{AB}{AD} = \frac{AC}{AE}$.

NOTE. This corollary is of importance in proving relations in similar triangles.

295. Cor. 2. *Three or more parallel lines cut off proportional segments on any two transversals.*



Hint. Draw $DH \parallel AC$, $AB = DG$, $BC = GH$.

Ex. 1. In the diagram for Prop. XIII, if $AD = 4$, $DB = 8$, $AE = 3$, find EC.

Ex. 2. In the same diagram, find DB, if $AD = a$, $AE = b$, and $EC = c$.

Ex. 3. In the same diagram, find AE, if $AB = 12$, $AD = 8$, and $AC = 9$.

Ex. 4. In the same diagram, find AE, if $AB = m$, $AD = n$, and $AC = p$.

Ex. 5. In the same diagram, find AD, if $AD = EC$, $DB = 4$, and $AE = 9$.

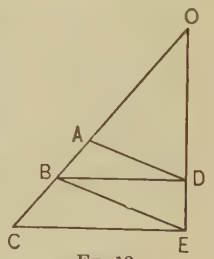
Ex. 6. In the same diagram, find AE, if $AE = 2\,DB$, $AD = 10$, and $EC = 20$.

Ex. 7. In the same diagram, find EC if $EC = AD$, $AE = m$, and $BD = n$.

Ex. 8. In the same diagram, find EC, if $AB = a$, $AD = b$, and $AC = c$.

Ex. 9. In the same diagram if $DE \parallel BC$, and $AD : DB = EC : AE$, then $AE = EC$.

Ex. 10. In the same diagram, if $AD = 2(AE)$ and $DB = 6$, find EC.



Ex. 12

Ex. 11. If in the diagram for Prop. XIII, $AD = 2$, $DB = 3$, $AE = 4$, and $EC = 4$, is DE parallel to BC?

Ex. 12. In the diagram, if $BD \parallel CE$, and $AD \parallel BE$, then

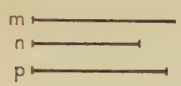
$$OA : OB = OB : OC.$$

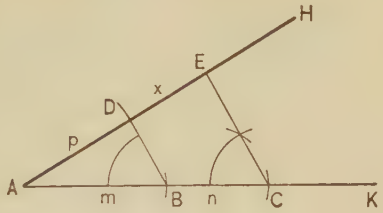
Ex. 13. In diagram for Prop. XIII, find DB and EC, if $AD = 8$, $DB = AE$, and $AC = 6$.

Ex. 14. In diagram for Prop. XIII, find DB and AE, if $AD = 8$, $AE =$ one half DB, and $EC = 1$.

Proposition XIV. Problem

296. *To find the fourth proportional to three given lines.*





Given three lines m, n, and p.

Required the fourth proportional to m, n, and p.

Construction. Draw any angle KAH.

On AK, make $AB = m$, $BC = n$; on AH, make $AD = p$.

Draw BD.

Through C, draw a line parallel to BD, meeting AH in E.

DE is the required fourth proportional.

[The proof is left to the student. Hint. Apply Prop. XIII.]

Ex. 1. If a, b, and c are given lines, construct a line x, so that $a : b = x : c$.

Ex. 2. If a, b, and c are given lines, construct a line equal to $\frac{bc}{a}$.

Ex. 3. If a, b, and c are given lines, construct a line equal to $\frac{bc}{2a}$.

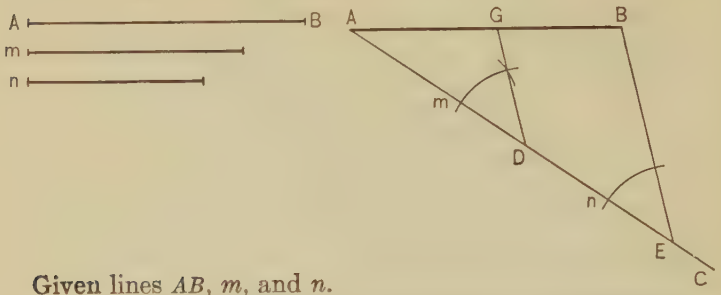
Ex. 4. If a and b are two given lines, construct a line $x = \frac{b^2}{a}$; to $\frac{2b^2}{3a}$.

Ex. 5. Construct $x = a(a + b) \div b$.

Ex. 6. Construct the fourth proportional to three lines $\frac{7}{8}''$, $\frac{3}{4}''$, and $1''$ long.

Proposition XV. Problem

297. *To divide a given line into segments proportional to two given lines.*



Given lines AB, m, and n.

Required to divide AB into segments proportional to m and n.

Construction. At any angle with AB draw AC.

On AC, cut off $AD = m$, $DE = n$.

Draw EB.

Through D draw a parallel to BE, intersecting AB in G.

Then AB is divided as required.

[The proof is left to the student. Hint. Apply Prop. XIII.]

Ex. **1**. Divide a given line AC into two segments having the ratio of 2 : 3.

Ex. **2**. Divide a 2″ line into two segments having the ratio that a diagonal of a square has to one of its sides.

Ex. **3**. Divide a given line into 3 segments having the continued ratio of three lines s, t, and w.

Ex. **4**. Construct two lines when their sum and their ratio are given.

Ex. **5**. Construct two lines when their difference and their ratio are given.

Ex. **6**. In a given line, AB, to find a point, C, so that

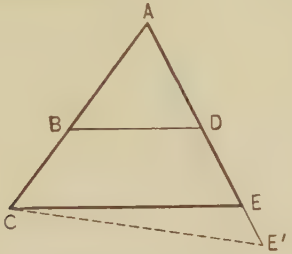
$$AB : AC = m : n,$$

when m and n are two given lines.

Proposition XVI. Theorem

298. *If a line divides two sides of a triangle into segments that are proportional, it is parallel to the third side.*

[Converse to Prop. XIII.]



Given in $\triangle AEC$, $AB : BC = AD : DE$.

To prove DB parallel to EC.

Proof.

1.	Through C, draw $CE' \parallel BD$, meeting AE in E'.	(Why?)
2.	$AB : BC = AD : DE$.	(Hyp.)
3.	$AB : BC = AD : DE'$.	(292)
4.	$\therefore DE = DE'$.	(275)
5.	Point E coincides with point E'.	(Why?)
6.	$\therefore CE$ and CE' coincide.	(Why?)
7.	$\therefore BD \parallel CE$.	(Why?)

299. Cor. *If a line divides two sides of a triangle so that one side is to one of its segments as the second side is to its corresponding segment, then the line is parallel to the third side.*

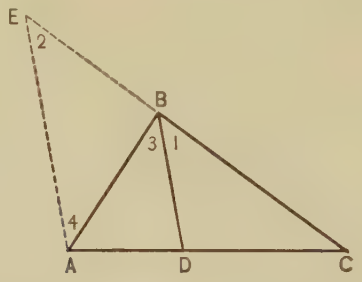
[Converse to Prop. XIII, Cor. 1.]

300. Remark. This corollary is often stated: "If a line divides two sides of a triangle proportionally, it is parallel to the third side."

Ex. In the diagram for Prop. XVI, if $AB = 12$, $BC = 16$, $AD = 15$, $DE = 20$, is $BD \parallel CE$?

Proposition XVII. Theorem

301. *The bisector of an angle of a triangle divides the opposite side into segments proportional to the adjacent sides.*



Given in $\triangle ABC$, BD bisecting $\angle ABC$.

To prove $AB : BC = AD : DC$.

Proof.

1.	Draw $AE \parallel DB$, to meet CB, produced, in E.	(Why?)
2.	$\angle 1 = \angle 2$.	(106)
3.	$\angle 3 = \angle 4$.	(104)
4.	$\angle 1 = \angle 3$.	(Hyp.)
5.	$\angle 2 = \angle 4$.	(Ax. 1.)
6.	$AB = BE$.	(121)
7.	$EB : BC = AD : DC$.	(292)
8.	$\therefore AB : BC = AD : DC$.	(Sub.)

302. Theorem. *The bisector of an exterior angle of a triangle divides the opposite side externally into segments proportional to the adjacent sides.*

Note. Show that this theorem has no meaning if the bisector of the exterior angle is at the vertex of an isosceles triangle.

Ex. 1. In the diagram for Prop. XVII, find DC, if $AB = 3$, $BC = 4$, and $AD = 2$.

Ex. 2. In the same diagram, find AD, if $AB = m$, $BC = n$, and $DC = p$.

Ex. 3. In the same diagram, find DC, if

(a) $AB = 4$, $BC = 5$, $AC = 6$.
(b) $AB = 18$, $BC = 9$, $AC = 21$.
(c) $AB = 21$, $BC = 14$, $AC = 25$.

Ex. 4. In the same diagram, find AD and DC, if $BC = a$, $CA = b$, and $AB = c$.

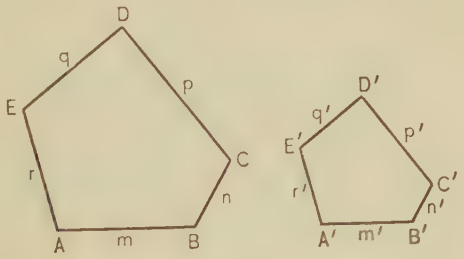
Ex. 5. In the same diagram, find AB, if $AB = DC$, $AD = 4$, and $BC = 16$.

Ex. 6. *To divide one side of a triangle into segments proportional to the other two sides.*

Ex. 7. State and prove the converse of Prop. XVII.

SIMILAR POLYGONS

303. Def. Two polygons are **similar** if their corresponding angles are equal and their corresponding sides are proportional.



Thus, polygons $ABCDE$ and $A'B'C'D'E'$ are similar, if

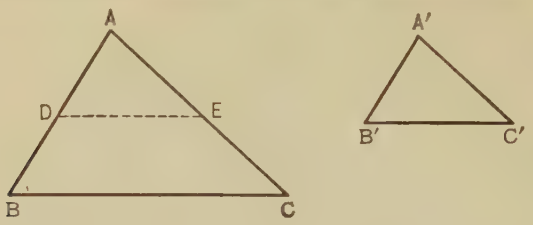
(1) $\angle A = \angle A'$, $\angle B = \angle B'$, $\angle C = \angle C'$, etc.

(2) $\frac{m}{m'} = \frac{n}{n'} = \frac{p}{p'}$, etc.

Ex. In two similar rectangles the bases are 12″ and 8″ respectively. If the altitude of the first is 9″ what is the altitude of the second rectangle?

Proposition XVIII. Theorem

304. *Two triangles are similar if the three angles of one are equal respectively to the three angles of the other.*



Given in $\triangle ABC$ and $A'B'C'$,

$$\angle A = \angle A',\ \angle B = \angle B', \text{ and } \angle C = \angle C'.$$

To prove $\triangle ABC \sim \triangle A'B'C'$.

Proof

STATEMENTS	REASONS
1. Place $\triangle A'B'C'$ upon $\triangle ABC$, $\angle A'$ on $\angle A$, taking the position $\triangle ADE$.	1. Why?
2. $\angle ADE = \angle B' = \angle B$.	2. Hyp.
3. $DE \parallel BC$.	3. If 2 cor. $\measuredangle$ are $=$, lines are $\parallel$.
4. $AB : AD = AC : AE$.	4. (294)
5. $AB : A'B' = AC : A'C'$.	5. Sub.
6. By placing $\triangle A'B'C'$ on $\triangle ABC$ so that $\angle B'$ coincides with $\angle B$, we can show that $AB : A'B' = BC : B'C'$.	6. Steps similar to (1–5).
7. $\therefore \dfrac{AB}{A'B'} = \dfrac{BC}{B'C'} = \dfrac{AC}{A'C'}$.	7. Why?
8. But $\angle A = \angle A'$, $\angle B = \angle B'$, $\angle C = \angle C'$.	8. Hyp.
9. $\therefore \triangle ABC \sim \triangle A'B'C'$.	9. { Their corr. $\measuredangle$ are $=$. Their corr. sides are proportional.

305. Cor. 1. *Two triangles are similar if two angles of the one are equal respectively to two angles of the other.*

306. Cor. 2. *Two right triangles are similar if an acute angle of the one is equal to an acute angle of the other.*

307. Cor. 3. *A line parallel to one side of a triangle cuts off a triangle similar to the given triangle.*

308. Cor. 4. *If two triangles are similar to a third triangle, they are similar to each other.*

Ex. 1. The sides of a polygon are 3, 4, 5, 6, and 7. Find the sides of a similar polygon if the side corresponding with 3 equals 15.

Ex. 2. The sides of a triangle are a, b, and c. Find the sides of a similar triangle if the side corresponding to a is equal to m.

Ex. 3. If two chords AB and CD intersect in E, the triangle AEC is similar to the triangle BED.

Ex. 4. If from a point A without a circle, two secants are drawn to meet the circle in B and C, and D and E, respectively, the triangle ABE is similar to the triangle ACD.

Ex. 5. If the altitudes AD and BE of the triangle ABC intersect in F, the triangle AFE is similar to the triangle BFD.

Ex. 6. Prove: Equilateral triangles are similar.

Ex. 7. Prove: *Two isosceles triangles having equal vertex (or equal base) angles are similar.*

Ex. 8. Show two squares are always similar

Ex. 9. *The diagonals of a trapezoid divide each other into segments that are proportional.*

Ex. 10. If AD, the angle bisector of the inscribed triangle ABC, is produced to meet the circle in E, the triangle ABD is similar to the triangle AEC.

Ex. 11. Two circles are tangent internally. Prove that all chords of the larger circle drawn from the point of contact are divided in the same ratio by the smaller circle.

309. Method XVII. **Similar triangles are the usual means of proving that lines are proportional.** To prove, therefore, that four lines are proportional:

(1) *Select two triangles so that each contains two of the given lines.* (It is advisable to mark the lines as indicated in 254 and 255.)

(2) *Prove the similarity of the two triangles.* (If triangles are not similar, select another pair.)

(3) *Derive the proportion.*

(4) (*Apply alternation and inversion*, if necessary.)

Ex. 1. If in the triangle ABC the altitudes AD and BE be drawn, prove that $AC : BC = DC : EC$.

Ex. 2. In the same diagram, if AD and BE meet in F, prove $BF : FA = DF : FE$.

Ex. 3. In the same diagram, $AE : AD = FE : DC$.

Ex. 4. If from the vertex A of an inscribed triangle ABC the altitude AD and the diameter AF be drawn, then $AB : AD = AF : AC$.

Ex. 5. In the same diagram, $BD : FC = AD : AC$.

Ex. 6. *If from a point without a circle a tangent and a secant be drawn, the tangent is the mean proportional between the secant and its external segment.*

Ex. 7. If a diameter AB be produced to C, at C a perpendicular be erected, and through B a line be drawn to meet the circle and the perpendicular in D and E respectively, then $AB : BE = DB : BC$.

Ex. 8. If in a right triangle ABC the altitude AD be drawn upon the hypotenuse, $AD : AB = AC : BC$.

Ex. 9. In the same diagram, $AD : AB = DC : AC$.

Ex. 10. If the angle-bisector CD of an inscribed triangle ABC be produced to meet the circle in E, $EB : EC = DB : CB$.

Ex. 11. If CD, the angle-bisector of the inscribed triangle ABC, be produced to meet the circle in E, then EB is a mean proportional between CE and DE.

Ex. 12. In the same diagram, $AD : EB = AC : CE$.

Ex. 13. *In similar triangles corresponding angle bisectors have the same ratio as any two corresponding sides.*

Ex. 14. *In similar triangles corresponding altitudes have the same ratio as any two corresponding sides.*

310. METHOD XVIII. **To prove that the product of two lines equals the product of two other lines,** *use the method of* (309) *and take the products of the means and extremes of the resulting proportion.*

Ex. 1. *If two chords intersect within a circle, the product of the segments of one equals the product of the segments of the other.*

Ex. 2. If from any point E in the chord AB the perpendicular EC be drawn upon the diameter AD, then $AC \times AD = AB \times AE$.

Ex. 3. *The product of two arms of a right triangle is equal to the product of the hypotenuse and the altitude upon the hypotenuse.*

Ex. 4. If in triangle ABC, a parallel to AB meets BC and CA in E and D respectively, then $AC \times DE = DC \times AB$.

Ex. 5. The product of any altitude of a triangle and its corresponding side is equal to the product of any other altitude and its corresponding side.

Ex. 6. The three sides of triangles are 14, 15, 13, and the altitude upon 14 equals 12. Find the other altitudes. (Compare Ex. 5.)

Ex. 7. If in triangle ABC the altitudes AD and BE meet in F, then $BF \times BE = BC \times BD$.

Ex. 8. If in the triangle ABC the altitudes AD and BE meet in F, then $BD \times DC = DF \times AD$.

Ex. 9. If AB is a diameter, BD the tangent at B, and DA meets the circle in E, then $\overline{AB}^2 = AE \times AD$.

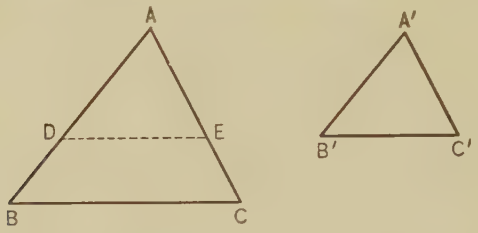
Ex. 10. In the same diagram, $\overline{BE}^2 = AE \times ED$.

Ex. 11. If the sides AB and DC of inscribed quadrilateral $ABCD$ are produced until they meet in E, and $\angle DBA = \angle CBE$, then $AD \times BE = CE \times BD$.

Ex. 12. If in $\triangle ABC$ the altitudes AD and BE are drawn and $BE = 6$, $EC = 3$, $DC = 2$, find AD.

Proposition XIX. Theorem

311. *Two triangles are similar if an angle of the one is equal to an angle of the other, and the sides including these angles are proportional.*



Given in $\triangle ABC$ and $A'B'C'$,
$\angle A = \angle A'$, and $AB : A'B' = AC : A'C'$.

To prove $\triangle ABC \sim \triangle A'B'C'$.

Proof

STATEMENTS	REASONS
1. Place $\triangle A'B'C'$ upon $\triangle ABC$ so that $\angle A'$ coincides with $\angle A$ and $\triangle ABC$ takes position $\triangle DAE$.	**1.** Why?
2. $AB : A'B' = AC : A'C'$.	**2.** Hyp.
3. $AB : AD = AC : AE$.	**3.** Sub.
4. $DE \parallel BC$.	**4.** Prop. XVI, cor.
5. $\angle B = \angle ADE$ and $\angle C = \angle DEA$.	**5.** Why?
6. $\therefore \triangle ABC \sim \triangle ADE$ (or $\triangle A'B'C'$).	**6.** Prop. XVIII.

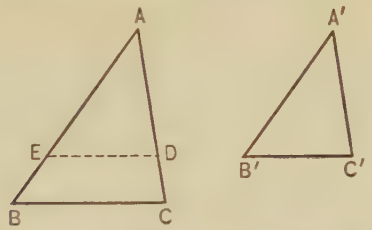
Ex. **1.** *In similar triangles, corresponding medians have the same ratio as any two corresponding sides.*

Ex. **2.** Two triangles, ABC and $A'B'C'$, are similar if altitude AD : altitude $A'D' = BC : B'C'$, and $\angle B = \angle B'$.

Ex. **3.** In similar triangles, the radii of the circumscribed circles have the same ratio as any two corresponding sides.

Proposition XX. Theorem

312. *Two triangles are similar if their corresponding sides are proportional.*



Given in $\triangle ABC$ and $A'B'C'$, $\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{AC}{A'C'}$.

To prove $\triangle ABC \sim \triangle A'B'C'$.

Proof

STATEMENTS	REASONS
1. On AC and AB respectively, lay off $AD = A'C'$ and $AE = A'B'$.	1. Why?
2. Draw DE.	2. Why?
3. $AB : A'B' = AC : A'C'$.	3. Hyp.
4. $AB : AE = AC : AD$.	4. Sub.
5. $\triangle ADE \sim \triangle ABC$.	5. (311).
6. $AB : AE = BC : ED$.	6. (303).
7. But $AB : A'B' = BC : B'C'$.	7. Hyp.
8. $ED = B'C'$.	8. (275).
9. $\therefore \triangle A'B'C' \cong \triangle ADE$.	9. (s. s. s. = s. s. s.).
10. $\therefore \triangle A'B'C' \sim \triangle ABC$.	10. Sub.

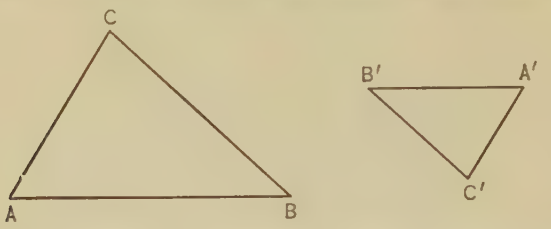
Ex. 1. The lines which join the mid-points of the sides of $\triangle ABC$ form a triangle similar to $\triangle ABC$.

Ex. 2. *Two triangles are similar if two sides and the median to one of these sides are proportional to the corresponding parts of the other.*

* Ex. 3. Two right triangles are similar if the hypotenuse and an arm of one triangle are proportional to the hypotenuse and an arm of the other.

Proposition XXI. Theorem

313. *Two triangles are similar if the sides of one are respectively parallel to the sides of the other.*



Given in $\triangle ABC$ and $A'B'C'$, $AB \parallel A'B'$, $AC \parallel A'C'$, and $BC \parallel B'C'$.

To prove $\triangle ABC \sim \triangle A'B'C'$.

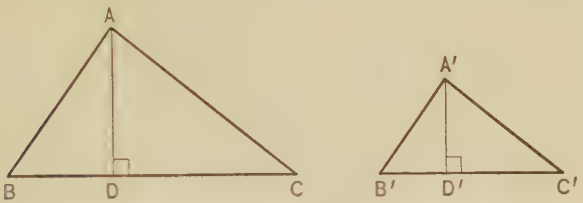
Proof

STATEMENTS	REASONS
1. Either $\angle A + \angle A' = 2$ rt. $\measuredangle$, or $\angle A = \angle A'$; $\angle B + \angle B' = 2$ rt. $\measuredangle$, or $\angle B = \angle B'$; $\angle C + \angle C' = 2$ rt. $\measuredangle$, or $\angle C = \angle C'$.	**1.** Why?
2. If $\angle A + \angle A' = 2$ rt. $\measuredangle$, $\angle B + \angle B' = 2$ rt. $\measuredangle$ and $\angle C + \angle C' = 2$ rt. $\measuredangle$, then $\angle A + \angle B + \angle C + \angle A' + \angle B' + \angle C' = 540°$.	**2.** Why?
3. This is impossible.	**3.** Sum of $\measuredangle$ of 2 $\triangle$ = 360°.
4. If $\angle A + \angle A' = 2$ rt. $\measuredangle$, $\angle B + \angle B' = 2$ rt. $\measuredangle$, and $\angle C = \angle C'$, then $\angle A + \angle B + \angle C + \angle A' + \angle B' + \angle C' = 360° + 2\angle C$.	**4.** Why?
5. This is impossible.	**5.** Same as 3.
6. $\therefore$ $\angle C = \angle C'$ and, say, $\angle B = \angle B'$.	**6.** Why?
7. $\therefore$ $\triangle ABC \sim \triangle A'B'C'$.	**7.** (305)

Ex. *Two triangles are similar if the sides of one are respectively perpendicular to the sides of the other.* Hint. Follow Prop. XXI.

Proposition XXII. Theorem

314. *If two triangles are similar, corresponding altitudes have the same ratio as any two corresponding sides.*



Given AD and $A'D'$, the corresponding altitudes of the similar triangles ABC and $A'B'C'$.

To prove $\dfrac{AD}{A'D'} = \dfrac{AB}{A'B'} = \dfrac{BC}{B'C'} = \dfrac{AC}{A'C'}$.

Proof

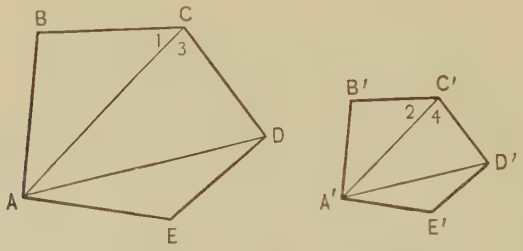
STATEMENTS	REASONS
1. $\triangle ABC \sim \triangle A'B'C'$.	1. Hyp.
2. $\angle BDA = \angle B'D'A'$.	2. Why?
3. $\angle B = \angle B'$.	3. Why?
4. Rt. $\triangle ABD \sim$ rt. $\triangle A'B'D'$.	4. (306).
5. $\dfrac{AD}{A'D'} = \dfrac{AB}{A'B'}$.	5. (303).
6. But $\dfrac{AB}{A'B'} = \dfrac{BC}{B'C'} = \dfrac{AC}{A'C'}$.	6. Why?
7. $\therefore$ $\dfrac{AD}{A'D'} = \dfrac{AB}{A'B'} = \dfrac{BC}{B'C'} = \dfrac{AC}{A'C'}$.	7. Why?

Ex. 1. The base of a triangle is 2 ft. and the altitude 9 in. If the corresponding base of a similar triangle is 6 in., find the corresponding altitude.

Ex. 2. Prove: In two similar triangles corresponding altitudes divide the triangles into similar triangles.

Proposition XXIII. Theorem

315. *Two similar polygons may be divided into the same number of similar triangles, similar each to each, and similarly placed.*



Given polygon $ABCDE \sim$ polygon $A'B'C'D'E'$.

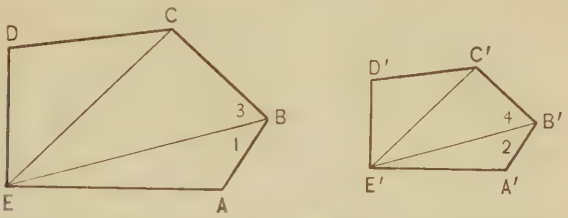
To prove $\triangle ABC \sim \triangle A'B'C'$,
$\triangle ACD \sim \triangle A'C'D'$, etc.

Proof

STATEMENTS	REASONS
1. $AB : A'B' = BC : B'C' = CD : C'D'$, etc. $\angle B = \angle B', \angle C = \angle C', \angle D = \angle D'$.	**1.** Why?
2. $\triangle ABC \sim \triangle A'B'C'$.	**2.** (311).
3. $\angle 1 = \angle 2$.	**3.** Why?
4. $\angle 3 = \angle 4$.	**4.** Why?
5. $\frac{BC}{B'C'} = \frac{CD}{C'D'}$.	**5.** Hyp.
6. $\frac{BC}{B'C'} = \frac{CA}{C'A'}$.	**6.** (303).
7. $\frac{CA}{C'A'} = \frac{CD}{C'D'}$.	**7.** Why?
8. $\triangle ACD \sim \triangle A'C'D'$.	**8.** (311).
9. Similarly we can prove $\triangle ADE \sim \triangle A'D'E'$, etc.	**9.** Steps (1–8).

Proposition XXIV. Theorem

316. *Two polygons are similar if they are composed of the same number of triangles, similar each to each, and similarly placed.*



Given in the polygons $ABCDE$ and $A'B'C'D'E'$, $\triangle ABE \sim \triangle A'B'E'$, $\triangle BCE \sim \triangle B'C'E'$, $\triangle CDE \sim \triangle C'D'E'$, etc., and similarly placed.

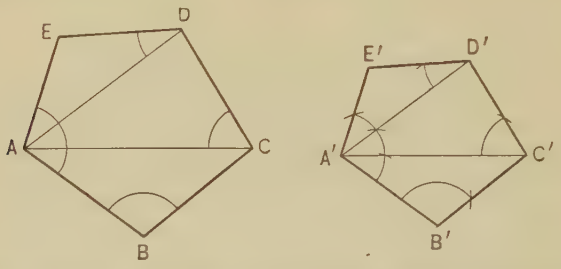
To prove polygon $ABCDE \sim$ polygon $A'B'C'D'E'$.

Proof

STATEMENTS	REASONS
1. $\triangle ABE \sim \triangle A'B'E'$.	**1.** Hyp.
2. $\frac{AB}{A'B'} = \frac{BE}{B'E'}$, $\angle 1 = \angle 2$.	**2.** Def. $\sim$ $\triangle$.
3. $\triangle EBC \sim \triangle E'B'C'$.	**3.** Hyp.
4. $\frac{BE}{B'E'} = \frac{BC}{B'C'}$, $\angle 3 = \angle 4$.	**4.** Same as 2.
5. $\frac{AB}{A'B'} = \frac{BC}{B'C'}$.	**5.** Why?
6. $\angle 1 + \angle 3 = \angle 2 + \angle 4$.	**6.** Why?
7. $\angle B = \angle B'$.	**7.** Sub.
8. Likewise, $\frac{BC}{B'C'} = \frac{CD}{C'D'}$, etc., and $\angle C = \angle C'$, $\angle D = \angle D'$, etc.	**8.** Steps (1–7).
9. $\therefore$ Polygon $ABCDE \sim$ polygon $A'B'C'D'E'$.	**9.** (303).

Proposition XXV. Problem

317. *To construct a polygon similar to a given polygon upon a line corresponding to a side of the given polygon.*



Given polygon $ABCDE$, and side $A'B'$ corres. to side AB.

Required to construct a polygon similar to $ABCDE$.

Construction. Draw all diagonals from one vertex, as A, in the given polygon. With $A'B'$ as a side and A' and B' as vertices construct $\angle B'A'C' = \angle BAC$ and $\angle B' = \angle B$. Extend sides to meet at C'. On $A'C'$ at A' and C' construct $\angle C'A'D' = \angle CAD$ and $\angle A'C'D' = \angle ACD$, etc.

Proof: $\triangle A'B'C' \sim \triangle ABC$, $\triangle C'A'D' \sim \triangle CAD$, etc. (Why?) Hence polygon $A'B'C'D'E' \sim ABCDE$. (Why?)

Ex. **1**. Draw polygons of 4, 5, and 6 sides. On a given line corresponding to a side in each of the polygons, construct polygons of 4, 5, and 6 sides similar to the given polygons.

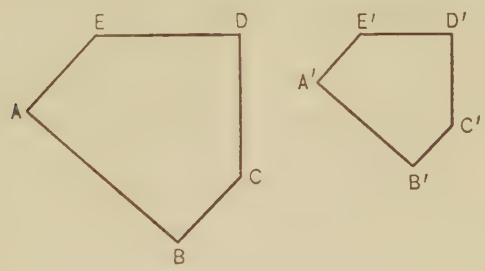
Ex. **2**. Draw a parallelogram. On AB as a base construct a similar parallelogram.

Ex. **3**. To construct a quadrilateral $ABCD$ similar to a given quadrilateral $A'B'C'D'$ and having the diagonal AC equal a given line.

Ex. **4**. Using Prop. XXV, show how to construct one rectangle similar to another rectangle.

Proposition XXVI. Theorem

318. *The perimeters of two similar polygons are to each other as any two corresponding sides.*



Given P and P', the perimeters of the similar polygons $ABCDE$ and $A'B'C'D'E'$ respectively.

To prove $P : P' = AB : A'B' = BC : B'C' = CD : C'D'$, etc.

Proof

	STATEMENTS	REASONS
1.	$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'} = \cdots$	1. Hyp.
2.	$\frac{AB + BC + CD + \cdots}{A'B' + B'C' + C'D' + \cdots} = \frac{AB}{A'B'} = \frac{BC}{B'C'} \cdots$.	2. (284).
3.	$\therefore \quad \frac{P}{P'} = \frac{AB}{A'B'} = \frac{BC}{B'C'} = \cdots$.	3. Sub.

Ex. 1. *The perimeters of two similar polygons are to each other as any two corresponding diagonals.*

Ex. 2. The perimeters of two similar triangles are to each other as any two corresponding altitudes.

Ex. 3. The sides of a polygon are 4, 5, 6, 7, 8 respectively. Find the perimeter of a similar polygon, if the side corresponding to 5 is 7.

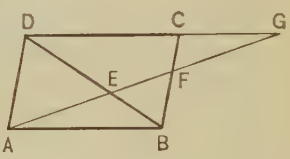
Ex. 4. The perimeters of two similar polygons are 20 and 25 inches, respectively. If a side of the first polygon is 4 in., find the corresponding side of the second polygon.

Ex. 5. In the diagram for Prop. XXVI find the perimeter of $ABCDE$, if the perimeter of $A'B'C'D'E'$ equals 20 inches, $A'B' = 4$ inches, $B'C' = 3$ inches, $AC = 10$ inches, $B'C' : A'B' = A'B' : A'C'$ and $ABCDE \sim A'B'C'D'E'$.

319. METHOD XIX. **To prove the proportionality of four lines which do not form similar triangles, find a third ratio equal to each of the given ones.**

Thus, in the annexed figure, if $ABCD$ is a parallelogram and AG is a straight line, then

$$\frac{EF}{EA} = \frac{EA}{EG}.$$



Obviously no two triangles exist, each of which contains two of the four lines. Hence we have to find a third ratio equal to each of the given ones. This ratio is $\frac{EB}{ED}$. Our problem is therefore split into two, viz.:

(*a*) To prove $\frac{EF}{EA} = \frac{EB}{ED}$, and

(*b*) To prove $\frac{EA}{EG} = \frac{EB}{ED}$.

Each of these proportions is easily proved by means of the fundamental method (XVII).

Ex. 1. *In similar triangles the radii of the inscribed circles have the same ratio as any two corresponding sides.*

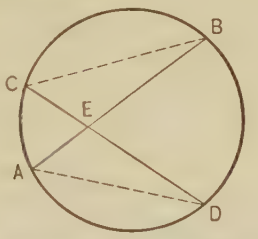
Ex. 2. If triangle ABC is similar to triangle $A'B'C'$, and AD and $A'D'$ are angle-bisectors, AF and $A'F'$ are altitudes, prove that $AD : A'D' = AF : A'F'$

Ex. 3. If in the similar triangles ABC and $A'B'C'$, the points D and D' are taken respectively in BC and $B'C'$, so that $\angle BAD = \angle B'A'D'$, then $BD : B'D' = BC : B'C'$.

Ex. 4. Prove that the $\triangle ADF$ and $A'D'F'$, in Ex. 2, are similar.

Proposition XXVII. Theorem

320. *If two chords intersect within a circle, the product of the segments of one is equal to the product of the segments of the other.*



Given in $\odot O$, the chords AB and CD intersecting in E.

To prove $AE \times EB = CE \times ED$.

Proof

STATEMENTS	REASONS
1. Draw CB and AD.	1. Why?
2. $\angle A = \angle C$, $\angle B = \angle D$.	2. Meas. by $\frac{1}{2}\overset{\frown}{BD}$ and $\frac{1}{2}\overset{\frown}{AC}$ resp.
3. $\triangle CEB \sim \triangle AED$.	3. Why?
4. $AE : CE = ED : EB$.	4. (303).
5. $\therefore AE \times EB = CE \times ED$.	5. (274).

321. Cor. *The product of the segments of a chord that passes through a fixed point within a circle is the same for all directions of the chord.*

Ex. 1. In the diagram for Prop. XXVII, if $AE = 3$, $EB = 4$, $ED = 6$, find CE.

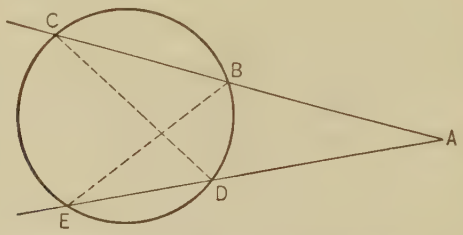
Ex. 2. In the same diagram, if $AE = a$, $EB = b$, and $ED = c$, find CE.

Ex. 3. In the same diagram, if $AE = 4$, $EB = 9$, and $CE = ED$, find CE.

Ex. 4. If the prolongations of two chords meet without a circle, is Prop. XXVII correct for the external segments of the chords?

Proposition XXVIII. Theorem

322. *If from a point without a circle, two secants are drawn, the product of one secant and its external segment is equal to the product of the other secant and its external segment.*



Given two secants AC and AE cutting a circle in B, C, D, and E respectively.

To prove $AC \times AB = AE \times AD$.

Proof

STATEMENTS	REASONS
1. Draw CD and EB.	**1.** Why?
2. $\angle C = \angle E$.	**2.** (232).
3. $\angle A = \angle A$.	**3.** Iden.
4. $\angle CDA = \angle EBA$.	**4.** Why?
5. $\triangle CDA \sim \triangle EBA$.	**5.** (304).
6. $AC : AE = AD : AB$.	**6.** (303).
7. $\therefore AC \times AB = AE \times AD$.	**7.** (274).

Ex. **1.** If, in the diagram for Prop. XXVIII, $CB = 16$, $BA = 2$, $DA = 4$, find DE.

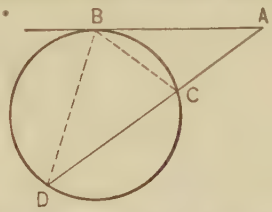
Ex. **2.** In the same diagram, if $CA : DA = 2 : 1$, prove that $EA = 2\,BA$.

Ex. **3.** If in the same diagram $CB = a$, $BA = b$, $AE = c$, find DA.

Ex. **4.** In the same diagram, if $AB = AD$, then $BCED$ is an isosceles trapezoid.

Proposition XXIX. Theorem

323. *If from a point without a circle, a tangent and a secant are drawn, the tangent is the mean proportional between the secant and its external segment.*



Given the tangent AB touching the $\odot BDC$ in B, and the secant AD cutting the circle in C and D.

To prove $AD : AB = AB : AC$.

Proof

STATEMENTS	REASONS
1. Draw BD and BC.	1. Why?
2. $\angle D = \angle CBA$.	2. Why?
3. $\angle A = \angle A$.	3. Iden.
4. $\angle DBA = \angle BCA$.	4. Why?
5. $\triangle DBA \sim \triangle BCA$.	5. (304).
6 $\therefore AD : AB = AB : AC$.	6. (303).

324. *If from a given point outside a circle, a secant is drawn, then the product of the whole secant times its external segment is constant, for all directions of the secant.*

Ex. 1. If in the diagram for Prop. XXIX, $DC = 5$, $AC = 4$, find AB.

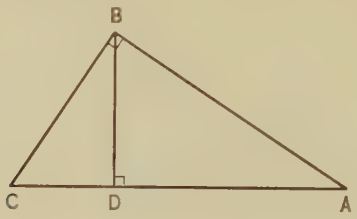
Ex. 2. In the same diagram, if $AB = 8$ inches, and the distance of A from the center of the circle equals 17 inches, find the radius of the circle.

Ex. 3. Tangents to two intersecting circles drawn from any point in the common chord, produced, are equal.

Ex. 4. If two circles are tangent externally and from any point in the common internal tangent secants are drawn to the two circles, the products of the secants and their external segments will be equal.

Proposition XXX. Theorem

325. *In any right triangle, the perpendicular from the vertex of the right angle on the hypotenuse divides the triangle into two triangles, similar to each other and to the given triangle.*



Given in the rt. $\triangle ACB$, BD the $\perp$ to hy. CA.

To prove $\triangle BCD \sim \triangle ACB \sim \triangle BDA$.

Proof

STATEMENTS	REASONS
1. Each $\triangle$ is a rt. $\triangle$.	1. Why?
2. In $\triangle ACB$ and BCD, $\angle C = \angle C$.	2. Iden.
3. $\therefore \triangle BCD \sim \triangle ACB$.	3. (306).
4. In $\triangle ACB$ and BDA, $\angle A = \angle A$.	4. Iden.
5. $\therefore \triangle ACB \sim \triangle BDA$.	5. (306).
6. $\therefore \triangle BCD \sim \triangle ACB \sim \triangle BDA$.	6. (308).

Ex. 1. In the figure for Prop. XXX, prove

$$CB \times BD = BA \times CD.$$

Ex. 2. In same figure if $CD = 9$ and $DA = 16$, find BD.

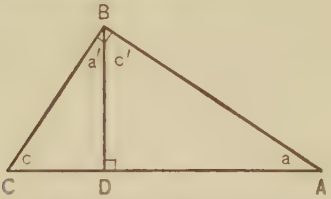
Ex. 3. Find BC and BA if $CD = 4$ and $DA = 9$. (Use fig. for Prop. XXX.)

Ex. 4. In the same figure, show that

$$BC \times BA = BD \times CA.$$

Proposition XXXI. Theorem

326. *In a right triangle, the altitude upon the hypotenuse is the mean proportional between the segments of the hypotenuse, and either arm is the mean proportional between the hypotenuse and the segment adjacent to that arm.*



Given in the rt. $\triangle ABC$, BD the altitude upon the hypotenuse AC.

To prove (1) $CD : DB = DB : DA$.
(2) $CA : BA = BA : DA$.
(3) $CA : CB = CB : CD$.

Proof

STATEMENTS	REASONS
1. $\triangle ABC \sim \triangle BCD \sim \triangle BDA$.	**1.** Prop. 30.
2. Since $\triangle BCD \sim \triangle BDA$, $\therefore CD : DB = DB : DA$.	**2.** (303).
3. Since $\triangle ABC \sim \triangle BDA$, $\therefore CA : BA = BA : DA$.	**3.** (303).
4. Since $\triangle ABC \sim \triangle BCD$, $\therefore CA : CB = CB : CD$.	**4.** (303).

NOTE. $\angle a$ or $\angle A = \angle a'$ and $\angle c$ or $\angle C = \angle c'$. (Why?)

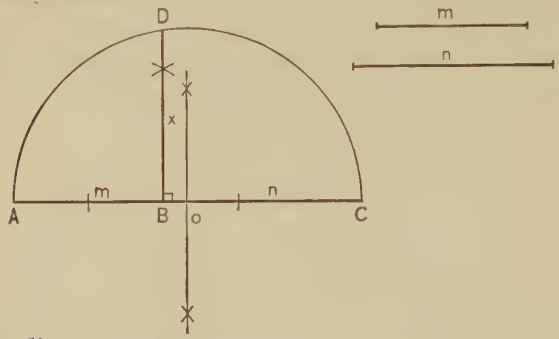
Ex. **1**. In the diagram for Prop. XXXI, if $AD = 4$ and $AC = 9$, what is the length of AB?

Ex. **2**. In the same diagram, if $AB = 15$, $AC = 25$, find AD and BD.

Ex. **3**. In the same diagram, if $BD = 24$, $AD = 18$, find AC and BC.

Proposition XXXII. Problem

327. *To construct the mean proportional between two given lines.*



Given lines m and n.

Required the mean proportional between m and n.

Construction. Draw $AB = m$, and produce AB to C so that $BC = n$.

On AC as a diameter, describe a semicircle.

At B, erect a perpendicular upon AC, meeting the circle in D.

BD is the required mean proportional.

[The proof is left to the student. Draw AD, DC. Apply Prop. XXXI.]

328. *The perpendicular from any point in the circle upon a diameter is the mean proportional between the segments of the diameter; and the chord joining the point to either extremity is the mean proportional between the diameter and the adjacent segment.*

Ex. 1. If a and b are given lines, construct $\sqrt{ab}$.

Ex. 2. Construct $\sqrt{6\,ab}$ if a and b are two given lines.

Ex. 3. Construct a line equal to $a\sqrt{2}$ if a is a given line.

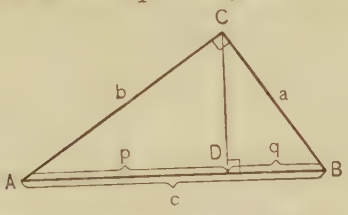
HINT. $a\sqrt{2} = \sqrt{(2\,a)\cdot a}$.

Ex. 4. Construct a line equal to $a\sqrt{5}$ if a is a given line.

Ex. 5. Construct a line equal to $a\sqrt{\frac{2}{3}}$ if a is a given line.

Proposition XXXIII. Theorem

329. *The sum of the squares of the arms of a right triangle is equal to the square of the hypotenuse.*



Given ABC a rt. $\triangle$, having its rt. $\angle$ at C.

To prove $a^2 + b^2 = c^2$.

Proof

STATEMENTS	REASONS
1. Draw $CD \perp AB$,	1. Why?
2. $p : b = b : c$.	2. (326).
3. $\therefore b^2 = c \cdot p$.	3. (274).
4. Similarly $a^2 = c \cdot q$.	4. Why?
5. $a^2 + b^2 = c \cdot p + c \cdot q$ $= c(p + q)$.	5. Why?
6. $a^2 + b^2 = c^2$.	6. (Sub.)

330. Cor. *The square of either arm of a right triangle is equal to the square of the hypotenuse, diminished by the square of the other arm.*

Ex. 1. Find the hypotenuse of a right triangle whose arms are respectively

(*a*) 1 ft. and 5 in. (*b*) m and n.

Ex. 2. The hypotenuse of a right triangle is 25, one arm equals 20; find the other arm.

Ex. 3. Find the altitude of an equilateral triangle whose side is equal to 8 in.

Ex. 4. *Find the altitude of an equilateral triangle whose side is equal to* b. $\left(h = \frac{b}{2}\sqrt{3}.\right)$

Ex. 5. Find the altitude of an isosceles triangle whose base equals 8 and whose arms equal 5.

Ex. 6. If the hypotenuse of an isosceles right triangle equals 8 in., what is the length of an arm?

Ex. 7. The radii of two circles are respectively 6 in. and 21 in., and the distance between their centers is 25 in.; find the length of the common external tangent.

Ex. 8. Find the side of an equilateral triangle whose altitude equals 10.

Ex. 9. The squares of the two arms of a right triangle have the same ratio as the segments adjacent to the hypotenuse.

Ex. 10. If AD is an altitude of a triangle ABC,

$$\overline{AB}^2 - \overline{AC}^2 = \overline{BD}^2 - \overline{CD}^2.$$

Ex. 11. If the diagonals of a quadrilateral are perpendicular to each other, then the sum of the squares of two opposite sides is equal to the sum of the squares of the other two.

Ex. 12. *If the square of one side of a triangle is equal to the sum of the squares of the other two sides, the triangle is a right triangle.*

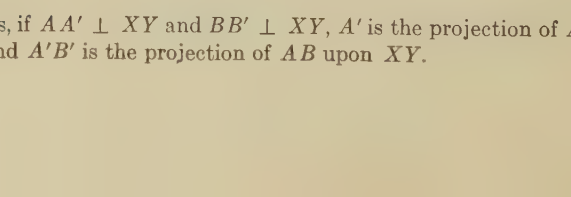
Hint. Draw a rt. $\triangle$ whose arms are equal respectively to the arms of the given $\triangle$, and prove the equality of the two $\triangle$.

*Ex. 13. If the square of one side of a triangle is greater than the sum of the squares of the other two, the triangle is obtuse.

Hint. Compare the $\triangle$ with a rt. $\triangle$ which has the same arms as the given one.

331. Def. The **projection of a point upon a line** is the foot of the perpendicular from the point to the line.

332. Def. The **projection of one line upon another** is the segment between the projections of the extremities of the first line upon the second.

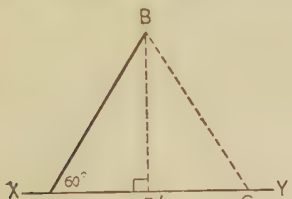


Thus, if $AA' \perp XY$ and $BB' \perp XY$, A' is the projection of A upon XY, and $A'B'$ is the projection of AB upon XY.

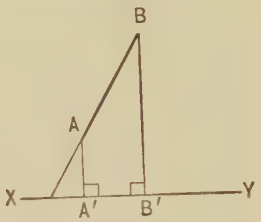
Ex. 1. In acute triangle ABC, draw the projection of AB upon AC, of AB upon BC, of AC upon AB.

Ex. 2. If $AB \parallel XY$, prove that the projection of AB upon XY equals AB.

Ex. 3. If the side of an equilateral triangle equals 10 in., what is the length of the projection of one side upon another?



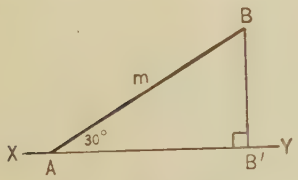
Ex. 4



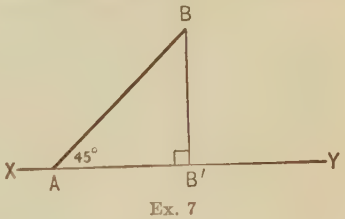
Ex. 5

Ex. 4. If the lines AB and XY include an angle of 60°, the projection of AB upon XY equals one half AB (165).

Ex. 5. Prove that the projection $A'B'$ of AB upon XY equals one half AB, if the *prolongation* of BA forms an angle of 60° with XY.



Ex. 6



Ex. 7

Ex. 6. If the lines AB and XY include an angle of 30°, and $AB = m$, prove that the projection of AB upon XY equals $\frac{m}{2}\sqrt{3}$.

Hint. $BB' = \frac{m}{2}$. (Ex. 4.)

Ex. 7. If the lines AB and XY include an angle of 45°, and $AB = m$, prove that the projection of AB upon $XY = \frac{m}{2}\sqrt{2}$.

Hint. If $AB' = x$, then $BB' = x$. $\therefore x^2 + x^2 = m^2$.

Ex. 8. Prove the last two exercises, if the *prolongation* of AB forms an angle of 30°; an angle of 45°. (Diagram similar to Ex. 5.)

Ex. 9. Find the projection of AB upon XY if $AB = m$, and the angle included by AB and XY equals 120°.

Ex. 10. Find the projection of AB upon XY if $AB = m$, and the angle included by AB and XY equals 135°.

Ex. 11. Find the projection of AB upon XY if $AB = m$, and the angle included by AB and XY equals 150°.

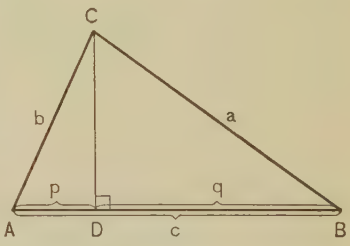
Ex. 12. If in triangle ABC, $AB = 8$, $AC = 10$, and $\angle A = 60°$, find the projection of AB upon AC, of BC upon AC.

Ex. 13. If in triangle ABC, $AB = 10$, $AC = 12$, and $\angle A = 45°$, find the projection of AB upon AC.

Ex. 14. In the same figure find the projection of BC upon AC.

Ex. 15. In triangle ABC, $AC = 24$, $BC = 10$, and $\angle C = 90°$. Find the projection of AC upon AB.

333. NOTE. $\triangle abc$ denotes a triangle whose sides are a, b, and c. p denotes the projection of b upon c, and q the projection of a upon c. The other notations used in the following propositions are in accordance with § 242.



Ex. 16. In $\triangle abc$, if $b = 4$, $p = 2$, find $\angle A$ and h_c.

Ex. 17. In $\triangle abc$, if $b = 5$, $h_c = 4$, $c = 8$, find q.

Ex. 18. In $\triangle abc$, if $b = 10$, $h_c = 8$, and $a = 17$, find c.

Ex. 19. In $\triangle abc$, if $b = 10$, $h_c = 8$, $c = 14$, find a.

Ex. 20. In $\triangle abc$, express a in terms of b, h_c, and c.

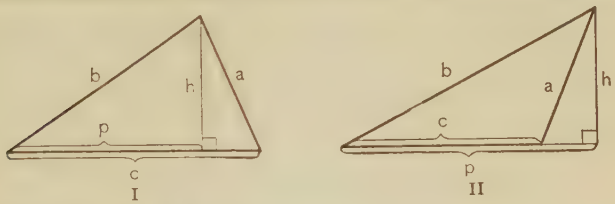
Ex. 21. In $\triangle abc$, if $a = 20$, $b = 37$, $q = 16$, find p.

Ex. 22. In $\triangle abc$, if $b = 15$, $p = 9$, and $c = 25$, find a.

Ex. 23. In $\triangle abc$, express a in terms of b, c, and p.

PROPOSITION XXXIV. THEOREM

334. *In any triangle, the square of a side opposite an* **acute** *angle is equal to the sum of the squares of the other two sides* **diminished** *by twice the product of one of those sides and the projection of the other side upon it.*



Given in $\triangle abc$, p the projection of b upon c, and the angle opposite a an acute angle.

To prove $a^2 = b^2 + c^2 - 2cp$.

Proof. Denote the perpendicular upon c by h.

In diagram I,

$$a^2 = h^2 + (c - p)^2,$$

but $$h^2 = b^2 - p^2.$$

[Substitute and simplify.]

In diagram II,

$$a^2 = h^2 + (p - c)^2,$$

but $$h^2 = b^2 - p^2.$$

[To be completed by the student.]

335. REMARK. The equation $a^2 = b^2 + c^2 - 2cp$ contains four quantities. Therefore **any one of them may be found by algebraical methods if the other three are given.** (Similarly in the following propositions.)

EX. 1. In $\triangle abc$, find a if

(*a*) $b = 8, c = 5, p = 4$.
(*b*) $b = 24, c = 9, p = 12$.
(*c*) $b = 5, c = 6, p = 3$.
(*d*) $b = 13, c = 14, p = 12$.
(*e*) $b = 17, c = 9, p = 15$.

Ex. 2. If two sides of a triangle equal 15 and 25 respectively, and the projection of 15 upon 25 equals 9, what is the value of the third side?

Ex. 3. In $\triangle abc$, find a if

(a) $b = 10$, $c = 16$, $\angle A = 60°$. (d) $b = 48$, $c = 13$, $\angle A = 60°$.

(b) $b = 14$, $c = 30$, $\angle A = 60°$. (e) $b = 4$, $c = 3$, $\angle A = 30°$.

(c) $b = 9$, $c = 24$, $\angle A = 60°$. (f) $b = 2$, $c = 3$, $\angle A = 45°$.

Ex. 4. The sides of a triangle are 13, 14, and 15. Find the projection of 13 upon 14.

Ex. 5. The sides of a triangle are 5, 7, and 8. Find the projection of 8 upon 5.

Ex. 6. The sides of a triangle are 10, 17, 21. Find the projection of 10 upon 21.

Proposition XXXV. Theorem

336. *In any obtuse triangle, the square of the side opposite the* ***obtuse*** *angle is equal to the sum of the squares of the other two sides,* ***increased*** *by twice the product of one of these sides and the projection of the other side upon it.*

Given in $\triangle abc$, p the projection of b upon c, and the angle opposite a obtuse.

To prove $\quad a^2 = b^2 + c^2 + 2cp.$

Proof. $\quad a^2 = h^2 + (c + p)^2.$

But $\quad h^2 = b^2 - p^2.$

$\therefore\ a^2 = ?$

[To be completed by the student.]

Ex. 1. In $\triangle abc$, $b = 6$, $c = 10$, $p = 3$, and $\angle A$ is obtuse; find a.

Ex. 2. In $\triangle abc$, $b = 10$, $c = 9$, $p = 6$, and $\angle A$ is obtuse; find a.

Ex. 3. In $\triangle abc$, find a if

(a) $b = 3$, $c = 5$, $\angle A = 120°$. (c) $b = 16$, $c = 5$, $\angle A = 120°$.

(b) $b = 8$, $c = 7$, $\angle A = 120°$. (d) $b = 24$, $c = 11$, $\angle A = 120°$.

337. Remark. If we consider the projection of one side of a triangle upon another as positive when the projection lies on that line, but as negative when it lies on the prolongation, Props. XXXV and XXXIII become special cases of Prop. XXXIV, and we always have:

$$a^2 = b^2 + c^2 - 2cp.$$

To compute the projections of sides of a triangle whose angles are not known, always apply this equation. If the result is negative, the triangle is obtuse.

Ex. 1. In $\triangle abc$, $a = 20$, $b = 15$, and $c = 7$. Find the projection of b upon c. Is the triangle obtuse or acute?

Ex. 2. In $\triangle abc$, $a = 20$, $b = 15$, $c = 25$. Find the projection of b upon c. Is angle A obtuse or acute?

Ex. 3. The sides of a triangle are 4, 13, and 15. Find the projection of 13 upon 4.

Ex. 4. If the value of p obtained from the above formula equals zero, what does this result signify?

Ex. 5. In $\triangle abc$, $a = 15$, $b = 13$, $c = 14$. Find h_c.

Ex. 6. In $\triangle abc$, $a = 17$, $b = 10$, $c = 9$. Find h_c.

338. Cor. 1. In $\triangle abc$, if p denotes the projection of b upon c,

$$p = \frac{b^2 + c^2 - a^2}{2c}.$$

339. Cor. 2. If h_c denotes the altitude upon c,

$$h_c = \sqrt{b^2 - p^2} = \sqrt{b^2 - \left(\frac{b^2 + c^2 - a^2}{2c}\right)^2}.$$

This expression can be simplified by algebraical operations:

$$h_c^2 = b^2 - \left(\frac{b^2+c^2-a^2}{2c}\right)^2 = \left(b + \frac{b^2+c^2-a^2}{2c}\right)\left(b - \frac{b^2+c^2-a^2}{2c}\right)$$

$$= \frac{(2bc + b^2 + c^2 - a^2)(2bc - b^2 - c^2 + a^2)}{4c^2}$$

$$= \frac{[(b+c)^2 - a^2][a^2 - (b-c)^2]}{4c^2}$$

$$= \frac{(a+b+c)(b+c-a)(a-b+c)(a+b-c)}{4c^2}.$$

Let $a + b + c = 2s$, *i.e.* let s denote half the perimeter.

$$b + c - a = 2(s - a).$$

$$a - b + c = 2(s - b).$$

$$a + b - c = 2(s - c).$$

$$h_c^2 = \frac{4s(s-a)(s-b)(s-c)}{c^2}$$

or

$$\boldsymbol{h_c = \frac{2}{c}\sqrt{s(s-a)(s-b)(s-c)}}.$$

Ex. **1.** In $\triangle abc$, find h_c, if

(*a*) $a = 10, b = 17, c = 21.$

(*b*) $a = 20, b = 13, c = 21.$

(*c*) $a = 20, b = 13, c = 25.$

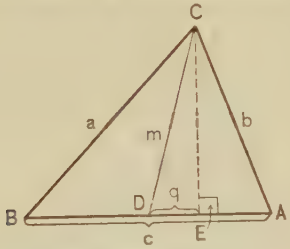
(*d*) $a = 37, b = 13, c = 40.$

Ex. **2.** The three sides of a triangle are 4, 13, and 15. Find the altitude upon 4.

Ex. **3.** The three sides of a triangle are 25, 30, and 11. Find the altitude upon 11.

Proposition XXXVI. Theorem

340. *In any triangle, the square of one side plus four times the square of the corresponding median is equal to twice the sum of the squares of the other sides.*



Given in $\triangle ABC$, m_c the median to c.

To prove $2\,a^2 + 2\,b^2 = c^2 + 4\,m_c^2$.

Proof. Draw $CE \perp AB$, and suppose E to fall between A and D. Let $DE = q$.

$$a^2 = \left(\frac{c}{2}\right)^2 + m^2 + 2\left(\frac{c}{2}\right)q. \qquad (336)$$

$$b^2 = \left(\frac{c}{2}\right)^2 + m^2 - 2\left(\frac{c}{2}\right)q. \qquad (334)$$

$$a^2 + b^2 = 2\left(\frac{c}{2}\right)^2 + 2\,m^2, \qquad (\text{Ax. } 2.)$$

or $$2\,a^2 + 2\,b^2 = c^2 + 4\,m^2. \qquad (\text{Ax. } 7.)$$

Ex. **1.** The sides of a triangle are 7, 8, and 9 respectively. Find the length of the median to 8.

Ex. **2.** The sides of a triangle are 7, 4, and 9 respectively. Find the length of the median to 9.

Ex. **3.** The sides of a triangle are 10, 5, and 9 respectively. Find the length of the median to 9.

Ex. **4.** The sides of a triangle are 22, 20, and 18 respectively. Find the length of the median to 18.

Ex. **5.** In $\triangle abc$, if m_c denotes the median drawn to c, prove that

$$m_c = \frac{1}{2}\sqrt{2\,a^2 + 2\,b^2 - c^2}.$$

Ex. 6. In $\triangle abc$, $a = 8$, $b = 11$, and $m_c = 8\frac{1}{2}$. Find c.

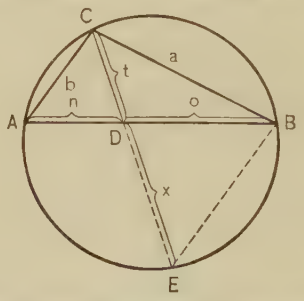
Ex. 7. In $\triangle abc$, $a = 28$, $c = 32$, $m_c = 38$. Find b.

Ex. 8. Prove Prop. XXXVI if $a = b$.

Ex. 9. *The sum of the squares of the four sides of a parallelogram is equal to the sum of the squares of the diagonals.*

Proposition XXXVII. Theorem

341. *In any triangle, the product of two sides is equal to the square of the bisector of the included angle plus the product of the segments of the third side.*



Given $\triangle abc$, the angle-bisector t dividing c into the segments n and o.

To prove $ab = t^2 + on.$

Proof. Circumscribe a circle about $\triangle abc$.

Produce the angle-bisector CD to meet the circle in E. Draw EB, and let $DE = x$.

$\angle ACD = \angle ECB.$ (Hyp.)

$\angle A = \angle E.$ (232)

$\triangle ACD \sim \triangle EBC.$ (305)

$\therefore b : t + x = t : a,$

or $ab = t(t + x).$

$ab = t^2 + tx.$

But $tx = on.$ (320)

$\therefore ab = t^2 + on.$

Ex. 1. The sides of a triangle are 18, 9, and 21 respectively. Find the length of the bisector corresponding with 21.

HINT. Find n and o by means of (301).

Ex. 2. The sides of a triangle are 21, 14, and 25 respectively. Find the length of the bisector corresponding with 25.

Ex. 3. The sides of a triangle are 22, 11, and 21 respectively. Find the length of the bisector corresponding with 21.

Ex. 4. The sides of a triangle are 6, 3, and 7 respectively. Find the length of the bisector corresponding with 7.

Ex. 5. In $\triangle abc$, if t denotes the angle bisector of $\angle C$, prove that

$$t^2 = ab - \frac{abc^2}{(a+b)^2}.$$

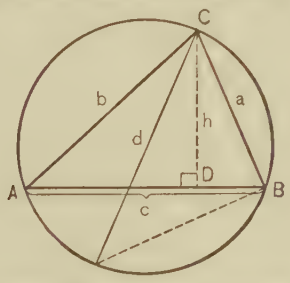
HINT. $$n = \frac{bc}{a+b},\ o = \frac{ac}{a+b}. \qquad (301)$$

* Ex. 6. Using the notation and the method of (339), reduce the result of the preceding exercise to the following form

$$t = \frac{2}{a+b}\sqrt{abs(s-c)}.$$

PROPOSITION XXXVIII. THEOREM

342. *In any triangle the product of two sides is equal to the altitude upon the third side, multiplied by the diameter of the circumscribed circle.*



Given d, a diameter of the circle circumscribed about $\triangle abc$, and h, the altitude upon c.

To prove $ab = hd$.

[The proof is left to the student.]

343. Cor. The diameter of the circumscribed circle of any triangle is equal to the product of two sides divided by the altitude upon the third side.

$$\left(d = \frac{ab}{h} \text{ or } d = \frac{abc}{2\sqrt{s(s-a)(s-b)(s-c)}}.\right)$$

Ex. 1. In $\triangle abc$, $a = 6$, $b = 10$, and $h_c = 4$. Find the diameter of the circumscribed circle.

Ex. 2. In $\triangle abc$, $a = 10$, $b = 15$, $h_c = 6$. Find the radius of the circumscribed circle.

Ex. 3. $\triangle ABC$ is inscribed in a circle of radius = 5 inches. Find the altitude to BC if $AB = 4$ and $AC = 5$.

Ex. 4. Find the diameter of the circle circumscribed about $\triangle abc$, if

(*a*) $a = 17$, $b = 8$, $c = 15$.

(*b*) $a = 10$, $b = 17$, $c = 21$.

Ex. 5. In $\triangle abc$, $a = 20$, $b = 15$, and the projection of b upon c equals 9. Find the radius of the circumscribed circle.

Ex. 6. In $\triangle abc$, $a = 9$ and $b = 12$. Find c if the diameter of the circumscribed circle equals 15.

PROBLEMS OF COMPUTATION

Ex. 1. The arms of a right triangle are 8 and 15 respectively. Compute the hypotenuse and the altitude upon the hypotenuse.

Ex. 2. A chord 14 in. long is 12 in. distant from the center. Find the radius of the circle.

Ex. 3. A chord 24 in. long is 5 in. distant from the center. Find the distance of the center from a chord 10 in. long.

Ex. 4. In $\triangle abc$, $a = 4$, $b = 8$, and the angle opposite to c equals 60°. Find c.

Ex. 5. The shadow of a church steeple upon level ground is 60 ft., while a pole 10 ft. high casts a shadow 3 ft. long. How high is the steeple?

Ex. 6. Find the product of the segments of a chord drawn through a point 8 in. from the center of a circle whose radius is 10 in. What is the length of the shortest chord that can be drawn through that point?

Ex. 7. The base of an isosceles triangle is 48 in. Find the altitude if each arm equals 50 in.

Ex. 8. Two sides and a diagonal of a parallelogram are 7, 9, and 8 respectively. Find the length of the other diagonal.

Ex. 9. The diagonal of a square is 20 in. Find the side.

Ex. 10. The sides of a rectangle are 16 and 30 respectively. Find the diagonal.

Ex. 11. The diameter AB of a circle is produced to C, and from C a tangent is drawn to the circle. Find the length of the tangent if $AB = 30$ and $BC = 2$.

Ex. 12. A ladder 17 ft. long reaches a window 15 ft. high. How far is the lower end of the ladder from the house?

Ex. 13. In $\triangle abc$, $c = 10$, $\angle A = 30°$, and $\angle C = 90°$. Find a and b.

Ex. 14. The base of an isosceles triangle is b, and each arm a. Find the altitude.

Ex. 15. The nonparallel sides AB and CD of a trapezoid are produced till they meet in E. Find AE and BE if $AB = 7$ and the bases are 5 and 3 respectively.

Ex. 16. The diagonals of a rhombus are 10 and 24 in. respectively. Find the perimeter; also, the altitude.

Ex. 17. Two parallel chords 1 in. apart are respectively 6 in. and 8 in. in length. Find the radius of the circle. (Is there more than one solution?)

Ex. 18. The altitude of a trapezoid is h, the bases a and b respectively. Find the altitudes of the two triangles formed by producing the nonparallel sides until they meet.

Ex. 19. From a point 24 ft. above sea level the visible horizon has a radius of 6 miles. Find the diameter of the earth.

Ex. 20. Find the length of the common internal tangent of two circles if the distance between the centers is 17, and the radii are 5 and 3 respectively.

Ex. 21. A chord 30 in. long subtends an angle of 120°. Find the distance of the chord from the center of the circle.

* Ex. 22. In triangle ABC, $AB = BC = 25$, $AC = 30$, and on AB is laid off $AD = 8$. Find the length of CD.

* Ex. 23. The three sides of a triangle are 14, 16, and 6. Find the angle opposite 14.

* Ex. 24. In a quadrilateral $ABCD$, $AB = 10$, $BC = 17$, $CD = 13$, $DA = 20$, and $AC = 21$. Find the diagonal BD.

PROBLEMS OF CONSTRUCTION

Ex. 1. Divide each side of a triangle internally into two parts proportional to the other two sides.

To construct a triangle, having given:

Ex. 2. a, b, and $b : c = 4 : 5$.

Ex. 3. From a given rectangle to cut off a similar rectangle by a line parallel to one of its sides.

Ex. 4. In a given circle, to inscribe a triangle similar to a given triangle.

Ex. 5. About a given circle, to circumscribe a triangle similar to a given triangle.

Ex. 6. Construct a triangle similar to a given triangle and having a given altitude.

Ex. 7. *To inscribe a square in a given triangle.*

Ex. 8. Assuming an arbitrary unit, construct a line equal to (*a*) $\sqrt{2}$, (*b*) $\sqrt{3}$, (*c*) $1 + \sqrt{5}$.

Ex. 9. Construct a line that shall be to a given line as $1 : \sqrt{2}$.

Ex. 10. Construct a line that shall be to a given line as $\sqrt{3} : 1$.

Ex. 11. In a given line AB, to find a point C such that $AC : BC = 1 : \sqrt{2}$.

Ex. 12. To construct a parallelogram similar to a given parallelogram and having a given diagonal.

Ex. 13. *To construct a triangle similar to a given triangle and having a given perimeter.*

Ex. 14. If a and b are two given lines, construct a line equal to $\frac{2\,a^2}{b}$.

Ex. 15. From a point without a circle, to draw a secant whose external segment is equal to one half the secant. (See hint for Ex. 16.)

Ex. 16. To construct a circle, touching a given circle in a given point, and touching a given line. HINT. Draw a tangent to the circle.

Ex. 17. To construct a circle, touching two parallel lines and passing through a given point.

Ex. 18. In a given circle, to inscribe a rectangle, having given the ratio of two sides.

* Ex. 19. To divide a trapezoid into two similar trapezoids by a line parallel to the bases.

THEOREMS

Ex. **1**. If a chord is bisected by another, either segment of the first is a mean proportional between the segments of the other.

Ex. **2**. If in the triangle ABC the altitudes BD and AE meet in F, and $AB = BC$, then

$$BC : AF = BD : CD.$$

Ex. **3**. If between two parallel tangents, a third tangent is drawn, the radius is the mean proportional between the segments of the third tangent.

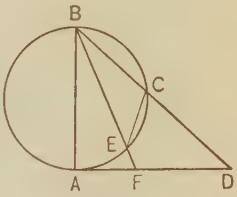
Ex. **4**. If two circles are tangent externally, and through the point of contact a secant is drawn, the chords formed are proportional to the radii.

Ex. **5**. If C is the mid-point of the arc AB, and a chord CD meets the chord AB in E, then

$$CE : CA = CA : CD.$$

Ex. **6**. If AB is a diameter, AD a tangent, and FB and DB are secants, prove that

$$BE \times BF = BC \times BD.$$



Ex. 6

Ex. **7**. Prove that in any right triangle the sum of the hypotenuse and the diameter of the inscribed circle is equal to the sum of the arms of the triangle.

Ex. **8**. If two circles intersect, their common chord produced bisects the common tangents.

Ex. **9**. If an isosceles triangle is inscribed in a circle, the tangents drawn at the vertices form another isosceles triangle.

Ex. **10**. The tangents drawn at the vertices of an inscribed rectangle inclose a rhombus.

Ex. **11**. *Two parallelograms are similar when they have an angle of the one equal to an angle of the other, and the including sides proportional.*

Ex. **12**. *Two rectangles are similar if two adjacent sides are proportional.*

Ex. **13**. Two circles are tangent externally, and through the point of contact two straight lines are drawn terminating in the circles; prove that the corresponding segments of the lines are proportional.

MISCELLANEOUS

Ex. 1. *If two parallel lines are cut by three or more transversals passing through a common point, the corresponding segments are proportional.*

Ex. 2. *Prove the converse of Ex. 1.*

Ex. 3. *The median to a side of a triangle bisects every line parallel to this side and limited by the other sides.*

Ex. 4. Construct an isosceles triangle given the base and the radius of the inscribed circle.

Ex. 5. Prove that the right angle of any right triangle is bisected by a line drawn from its vertex to the point of intersection of the diagonals of the square constructed on the hypotenuse.

Ex. 6. The bisectors of the equal angles of an isosceles triangle meet the equal sides at points D and E; prove that DE is parallel to the base of the triangle.

Ex. 7. Construct a right triangle, given the segments made on the hypotenuse by the bisector of the right angle.

Ex. 8. The perpendiculars from the mid-points of two sides of a triangle to the third side are equal.

Ex. 9. If two consecutive angles of a quadrilateral are right angles, the bisectors of the other angles are perpendicular to each other.

Ex. 10. Show how to construct a tangent to a circle at a given point on the circle, without using the center.

Ex. 11. The radius of a circle is 30 feet. Find the length of a tangent drawn to the circle from a point 4 feet from the circle.

Ex. 12. A rectangle is inscribed in a circle of radius 5 feet. The base is twice the altitude. Find the perimeter of the rectangle.

Ex. 13. Two circles of radii 12 in. and 9 in. respectively have their centers 28 in. apart. How far from the center of each circle is the line of centers cut by a common external tangent?

Ex. 14. The bases of a trapezoid are 8″ and 12″ respectively. The altitude is 3″. How long is the line drawn between the legs and parallel to the bases, if it is 2″ from the lower base?

BOOK IV

AREAS OF POLYGONS

344. DEF. The **unit of surface** is a square whose side is the unit of length, etc.

Thus, a square 1 in. long and 1 in. wide is a square inch. Or a square 1 yd. long and 1 yd. wide is a square yard.

345. The **area of a surface** is the number of units of surface it contains.

Thus, if the floor of a room is 25 ft. long and 15 ft. wide, it contains 15×25 or 375 sq. ft. Hence the area of the floor is 375 sq. ft.

346. Two figures are **equivalent** or **equal** if their areas are equal.

Thus, if the area of $\angle ABC$ = 25 sq. ft., and the area of $\square MNOP$ = 25 sq. ft., then $\triangle ABC$ is *equivalent to* $\square MNOP$, or in symbols:

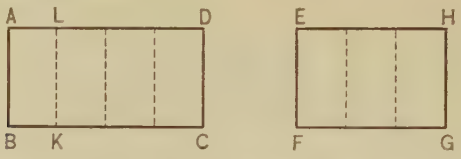
$$\triangle ABC = \square\, MNOP.$$

NOTE. The symbol = refers to the size of figures, while the symbol $\sim$ relates to their shape. The symbol of congruence $\cong$ relates to both size and shape. The symbol $\sim$ has no meaning for figures that cannot differ in shape, *e.g.* for straight lines. Thence the symbol of congruence, $\cong$ when applied to straight lines has the same significance as the symbol of equality = .

If the symbol of equality = refers to areas, it may read either "is equivalent to" or "equals." Since many authors, however, designate congruent figures as *equal figures*, confusion may be avoided by giving preference to the term equivalent. The use of a particular symbol for equivalence $\asymp$ cannot be recommended.

Proposition I. Theorem

347. *Rectangles having equal altitudes are to each other as their bases.*



Given rectangles $ABCD$ and $EFGH$ with bases BC and FG.

To prove $ABCD : EFGH = BC : FG$.

Case I (Commensurable)

Proof

STATEMENTS	REASONS
1. Take BK as unit of measure for the bases. Then $BC = m(BK)$, $FG = n(BK)$.	**1.** The bases are commensurable.
2. $\frac{BC}{FG} = \frac{m}{n}$.	**2.** (224).
3. At the points of division on BC and FG, erect $\perp$s. The rectangles will be cut up into m and n small rectangles, each equal to $ABKL$.	**3.** (47, 141, 147).
4. $\frac{ABCD}{EFGH} = \frac{m}{n}$.	**4.** (224).
5. $\therefore \frac{ABCD}{EFGH} = \frac{BC}{FG}$.	**5.** Why?

Case II. (Incommensurable.) See Appendix.

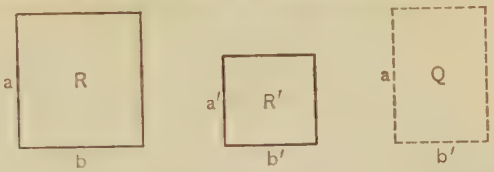
348. Cor. *Rectangles having equal bases are to each other as their altitudes.*

Ex. To divide a given rectangle into three congruent rectangles.

[Practical Applications, page 296.]

Proposition II. Theorem

349. *The areas of two rectangles are to each other as the products of their bases and altitudes.*



Given rectangles R and R', having the bases b and b' and the altitudes a and a' respectively.

To prove $$\frac{R}{R'} = \frac{a \times b}{a' \times b'}.$$

Proof

STATEMENTS	REASONS
1. Construct the rectangle Q, having the same base as R', and the same altitude as R.	1. To construct a rectangle, given base and altitude.
2. $\frac{R}{Q} = \frac{b}{b'}$.	2. Prop. I.
3. $\frac{Q}{R'} = \frac{a}{a'}$.	3. (348).
4. $\frac{R}{R'} = \frac{a \times b}{a' \times b'}$.	4. Why?

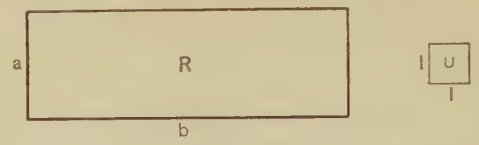
Ex. 1. Find the ratio of a rectangle 4 by 5 ft., and a square having a side of 10 ft.

Ex. 2. The diagonal of a rectangle is 26 in. long, and one of its sides 24 in. The diagonal of another rectangle is 25 in. long, and one of its sides 20 in. Find the ratio of the areas of the two rectangles.

Ex. 3. The ratio of the areas of two rectangles is as 8 is to 3. The dimensions of the first are 6 and 12. Find the altitude of the second rectangle, if its base is 9.

Proposition III. Theorem

350. *The area of a rectangle is equal to the product of its base and altitude.*



Given R a rectangle with base b and altitude a.

To prove Area of $R = a \times b$.

Proof

STATEMENTS	REASONS
1. Let U be the unit of surface.	1. Why?
2. $\frac{R}{U} = \frac{a \times b}{1 \times 1}$.	2. (349).
3. But $\frac{R}{U}$ is the area of R.	3. (345).
4. $\therefore$ area of $R = a \times b$.	4. Why?

351. Cor. *The area of a square is equal to the square of a side.*

352. Remark. If the base and altitude of a rectangle are expressed by integral numbers, Prop. III may be proved by dividing the figure into squares.

Thus, if the base contains 5 and the altitude 3 linear units, the figure can be divided into fifteen squares, each being the unit of surface.

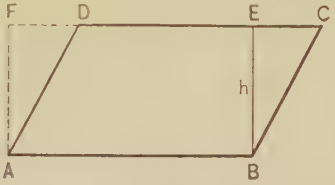
Ex. 1. A rectangular field is 24 yd. long and 15 yd. wide. Find the area.

Ex. 2. The area of a rectangle is 360 sq. in., and its base is 4 yd. Find the altitude.

Ex. 3. A rectangle has an area of 125 sq. in. and is five times as long as wide. Find the dimensions of the rectangle.

Proposition IV. Theorem

353. *The area of a parallelogram is equal to the product of its base and its altitude.*



Given parallelogram $ABCD$ having the base $AB = b$, and the altitude $BE = h$.

To prove $ABCD = b \times h$.

Proof

STATEMENTS	REASONS
1. Draw $AF \perp$ to AB to meet CD produced at F.	**1.** Why?
2. $AF \parallel BE$.	**2.** (96).
3. $ABEF$ is a rectangle, with base b and alt. h.	**3.** Why?
4. $ABCD$ is a $\square$ with base b and alt. h.	**4.** Hyp.
5. $AF = BE$, $AD = BC$.	**5.** Why?
6. Rt. $\triangle AFD \cong$ rt. $\triangle BEC$.	**6.** Hy. a. = hy. a.
7. $ABCF = ABCF$.	**7.** Iden.
8. $\square ABCD =$ Rect. $ABEF$.	**8.** Equals from equals ···.
9. But Rect. $ABEF = bh$.	**9.** Why?
10. $\therefore \square ABCD = bh$.	**10.** Sub.

354. Cor. 1. *Parallelograms having equal bases and equal altitudes are equivalent.*

355. Cor. 2. *Any two parallelograms are to each other as the products of their bases and altitudes.*

356. Cor. 3. *Parallelograms having equal bases are to each other as their altitudes.*

357. Cor. 4. *Parallelograms having equal altitudes are to each other as their bases.*

Ex. **1.** Find the area of a parallelogram whose base is 15 in. and whose altitude is 2 ft.

Ex. **2.** Two sides of a parallelogram are 15 and 20 respectively, and include an angle of 30°. Find the area.

Ex. **3.** The sides of a parallelogram are 5 and 8 respectively, and the projection of 5 upon 8 is 3. Find the area.

Ex. **4.** To divide a parallelogram into three equivalent parts.

Ex. **5.** The sides of a parallelogram are 13 and 14, and one diagonal equals 15. Find the area.

Ex. **6.** The sides of a parallelogram are 4 in. and 5 in. and the angle included by them equals 45°. Find the area.

Ex. **7.** The area of a parallelogram is 144 sq. inches. Find its altitude in inches, if the base is 3 feet long.

Ex. **8.** A parallelogram has its base equal to 30 feet and its altitude equal to 12 feet. Find the altitude of an equivalent rectangle having a base of 20 feet.

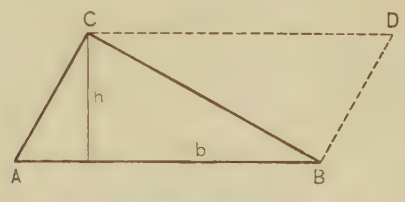
Ex. **9.** A rectangle is 12 by 20. Find the base and a side of an equivalent oblique parallelogram having an angle of 30°, if the base of the parallelogram is 4 units longer than a side.

Ex. **10.** The bases of two parallelograms are to each other as 3 is to 2. The area of the first is to that of the second as 5 is to 3. Compare their altitudes.

Ex. **11.** What change occurs in the area of a parallelogram: (*a*) when the base remains constant but the altitude increases (or decreases); (*b*) when the angle between the base and a side changes but the base and side remain constant?

PROPOSITION V. THEOREM

358. *The area of a triangle is equal to one half the product of its base and altitude.*



Given $\triangle ABC$, having base b and altitude h.

To prove $\triangle ABC = \frac{1}{2} b \times h$.

Proof

STATEMENTS	REASONS
1. Draw $BD \parallel AC$, and $CD \parallel AB$.	1. Why?
2. $ABDC$ is a $\square$.	2. Why?
3. $\square ABDC = bh$.	3. (353).
4. $\triangle ABC = \frac{1}{2} \square ABDC$.	4. (140).
5. $\frac{1}{2} \square ABDC = \frac{1}{2} bh$	5. Why?
6. $\therefore \triangle ABC = \frac{1}{2} bh$.	6. Why?

359. COR. 1. *Triangles having equal bases and equal altitudes are equivalent.*

360. COR. 2. *Any two triangles are to each other as the products of their bases and altitudes.*

361. COR. 3. *Triangles having equal bases are to each other as their altitudes.*

362. COR. 4. *Triangles having equal altitudes are to each other as their bases.*

Ex. 1. Prove: *If two triangles have a common base and their vertices lie in a line parallel to the base, the triangles are equivalent.*

Ex. 2. If A denotes the area, and $2s$ the perimeter of $\triangle abc$, then

$$A = \sqrt{s(s-a)(s-b)(s-c)}. \tag{339}$$

Ex. 3. The area of a triangle is 600 sq. in., and the altitude is 20 in. Find the base.

Ex. 4. Two sides of a triangle are 5 and 8 respectively, and include an angle of 30°. Find the area.

Ex. 5. Find the area of a triangle whose sides are respectively

(*a*) 13, 14, 15.
(*b*) 9, 10, 17.
(*c*) 11, 25, 30.
(*d*) 4, 13, 15.

Ex. 6. Find the area of $\triangle abc$, if $a = 10$, $b = 17$, $h_c = 8$. (2 sol.)

Ex. 7. *The diagonals of a parallelogram divide the parallelogram into four equivalent triangles.*

Ex. 8. Two triangles are equivalent if two sides of the one are equal respectively to two sides of the other, and the included angles are supplementary.

Ex. 9. What is the locus of the vertices of all equivalent triangles constructed on a common base?

Ex. 10. Find the altitude of a triangle, with a base of 8 in., that is equivalent to a parallelogram with a base of 14 in. and an altitude of 6 in.

Ex. 11. Find the area of an equilateral triangle having each side equal to 4 in.

Ex. 12. Find the area of an isosceles triangle whose base is 20 feet and whose vertex angle is 120°.

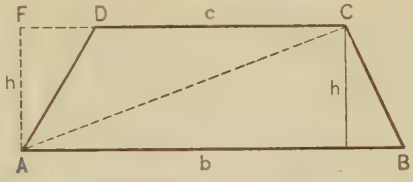
Ex. 13. Find the area of an isosceles right triangle having its hypotenuse equal to 18 yards.

Ex. 14. Find the area of an equilateral triangle if its altitude is $4\sqrt{3}$.

Ex. 15. What is the area of an isosceles triangle if an arm is 20 feet long and its base is 32 feet long?

Proposition VI. Theorem

363. *The area of a trapezoid is equal to one half the product of its altitude and the sum of its bases.*



Given. Trapezoid $ABCD$ has the bases b and c respectively, and the altitude h.

To prove $ABCD = \frac{1}{2}h(b + c)$.

Proof

STATEMENTS	REASONS
1. Draw AC; also, $AF \perp CD$ produced.	1. Why?
2. $AF = h$.	2. Why?
3. $\triangle ADC = \frac{1}{2}c \times AF$.	3. (358).
4. $\triangle ADC = \frac{1}{2}hc$.	4. Sub.
5. $\triangle ACB = \frac{1}{2}hb$.	5. (358).
6. $\therefore ABCD = \frac{1}{2}hb + \frac{1}{2}hc$, or $= \frac{1}{2}h(b + c)$.	6. Why?

Ex. 1. *The area of a trapezoid equals the product of its altitude and its median.*

Ex. 2. The line joining the mid-points of the bases of a trapezoid divides the trapezoid into two equivalent trapezoids.

Ex. 3. The bases of a trapezoid are 12 and 8 respectively, and the altitude is 5. Find the area.

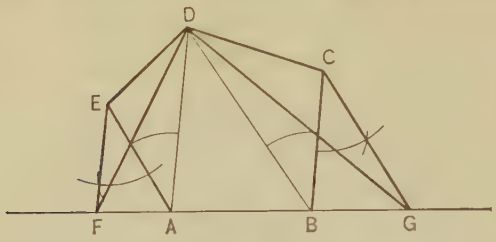
Ex. 4. In the diagram for Prop. VI, if $b = 6$, $c = 4$, $BC = 10$, and $\angle B = 30°$. Find the area.

Ex. 5. The area of a trapezoid is 200, the bases are 15 and 25 respectively. Find the altitude.

Ex. 6. The area of a trapezoid is 30, the altitude is 5, and one base is 8. Find the other base.

Proposition VII. Problem

364. *To transform a polygon into a triangle.*[1]



Given polygon *ABCDE*.

Required to construct a $\triangle$ equivalent to *ABCDE*.

Construction. Draw *DB* and *DA*. Through *C* and *E* draw *CG* and *EF* parallel respectively to *DB* and *DA*. Side *AB* produced will intersect these parallels at *F* and *G*. The required $\triangle$ is *FDG*.

Proof. $\triangle DBG = \triangle BCD$ and $\triangle DFA = \triangle DEA$. (Why?). Polygon $ABCDE = \triangle BCD + \triangle BDA + \triangle DEA = \triangle FDG$. (Why?)

Ex. **1**. Transform a square into a right triangle.

Ex. **2**. Transform a rectangle into a triangle.

Ex. **3**. Transform a quadrilateral into a triangle; also a trapezoid into a triangle.

Ex. **4**. Transform a parallelogram into a rectangle; into a right triangle.

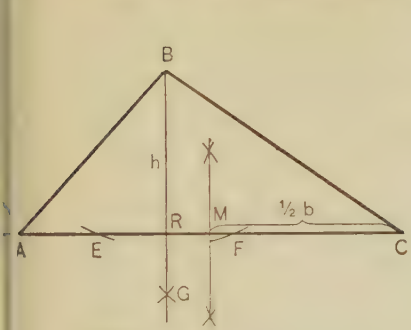
Ex. **5**. Transform an isosceles triangle into a rectangle; also an equilateral triangle into a rectangle.

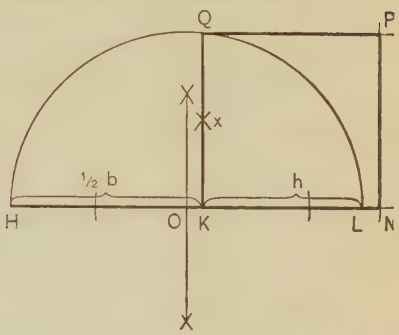
Ex. **6**. Transform a scalene triangle into an isosceles triangle.

[1] To transform one figure into another means to construct an equivalent figure.

PROPOSITION VIII. PROBLEM

365. *To transform a triangle into a square.*





Given $\triangle ABC$.

Required to construct a square equivalent to $\triangle ABC$.

Analysis. Let AC (the base) $= b$, $h =$ the alt. of $\triangle$. Area of $\triangle = \frac{1}{2}\,bh$. Let $x =$ a side of the required square. Area of square $= x^2$.

$\therefore\ x^2 = \frac{1}{2}\,bh$, or $x \cdot x = \frac{1}{2}\,b \cdot h$.

Then $\frac{1}{2}\,b : x = x : h$. Consequently, a side of the square is a mean proportional between the altitude and one half the base of the $\triangle$.

Construction. 1. Bisect base of triangle AC to get $MC = \frac{1}{2}\,b$.

2. Drop $\perp$ from B to base AC and call it h.

3. With $\frac{1}{2}\,b + h$ (or $HK + KL$) as a diameter of a circle construct a semicircle.

4. At the point (in diameter) common to $\frac{1}{2}\,b$ and h (i.e. K) construct $\perp$ to HL meeting semicircle in point Q. Then on QK (i.e. x) construct the square having x, or QK, as a side. [The proof lies in the analysis given above.]

Ex. Transform in equivalent squares:

(*a*) an equilateral $\triangle$
(*b*) an obtuse parallelogram
(*c*) a trapezoid
(*d*) a quadrilateral

366. METHOD XX. **One of the fundamental methods for transformation of geometric figures is the method that applies the algebraic equation.**

367. NOTE. In the following propositions, a, b, c, d, etc., denote given lines, while x, y, z, etc., denote required lines.

368. (1) Construct $x = a + b$.

(2) Construct $x = a - b$.

(3) If m denotes a given rational number.

Construct $x = m \cdot a$.

(4) Construct $x = \frac{a}{m}$.

(5) Construct $x = \frac{ab}{c}$.

HINT. x is the fourth proportional to c, a, and b.

(6) Construct $x = \frac{a^2}{b}$. [Use (5).]

(7) Construct $x = \sqrt{ab}$.

HINT. x is the mean proportional between a and b (327).

(8) Construct $x = \sqrt{a^2 + b^2}$.

HINT. x = hy. of rt. $\triangle$ with arms a and b.

(9) Construct $x = \sqrt{a^2 - b^2}$.

HINT. x = one arm of rt. $\triangle$ with hy. = a, arm = b.

(10) Construct $x = a\sqrt{2}$.

HINT. x = diag. of a square with side a.

(11) If m denotes a known number.

Construct $x = a\sqrt{m}$.

HINT. $x = \sqrt{a(am)}$.

Ex. 1. Construct x for each equality in section 368. [Let $m = 3$ in (3), (4), and (11).]

Ex. 2. Construct a square three times as large as a given square.

Ex. 3. Transform a parallelogram into a rectangle having its base twice that of the parallelogram.

Ex. 4. Construct a square equivalent to one half a given triangle.

Ex. 5. Construct $x = a\sqrt{5}$; $x = \frac{1}{2}a\sqrt{3}$; $x = \sqrt{a^2 + ab}$;

$$x = \sqrt{a^2 - ab}; \quad x = \frac{a^2 - b^2}{c}.$$

Ex. 6. Given the base and a base angle of a triangle, to construct the triangle equivalent to a given trapezoid.

Ex. 7. Construct a square equivalent to a given trapezoid.

Ex. 8. Given a rectangle, to construct an equivalent rectangle having its base equal to one half the perimeter of the given rectangle.

* Ex. 9. Construct an equilateral triangle equivalent to a given square.

369. The product of (the numerical measures of) two lines can be represented geometrically by the area of the rectangle formed by these lines. Hence, if the letters a, b, c, etc., represent a number of lines, then any homogeneous expression of the second degree that involves these letters can be represented by the sum (or difference) of rectangles.

NOTE. All diagrams of the following exercises may be constructed by drawing first two lines that are perpendicular to each other, and then constructing parallels to these according to the conditions of the theorem.

We may therefore assume in each exercise the following fact, which can easily be proved:

All quadrilaterals in the diagram are rectangles, and their opposite sides are equal.

Ex. If a, b, c, and d are four lines, $a > b$, and $b > c$ represent geometrically: (*a*) ab; (*b*) $(a + b)b$; (*c*) $(a - c)d$; (*d*) $(a + b)(c + d)$; (*e*) $(a - b)(c + d)$; (*f*) $ab + ac$; (*g*) $a^2 + ab + b^2$; (*h*) $a^2 - b^2$; (*i*) $(a - b)^2$.

370. Any homogeneous identity of the second degree can be demonstrated geometrically by showing that the areas which represent the two members of the identity are equal.

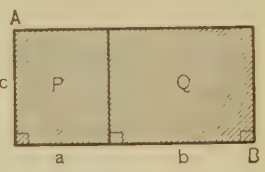
Thus to illustrate that

$(a + b)c = ac + bc$ we have,

$(a + b)c$ = area of AB.

ac = area of P.

bc = area of Q.



But obviously

$$AB = P + Q.$$

$$(a + b)c = ac + bc.$$

Similarly to show that

$(a - c)(b + d) = ab + ad - bc - bd$

$AH = a - c$, $AC = b + d$.

$\therefore$ the left member of the identity, or

$(a - c)(b + d)$ = area of AD.[1]

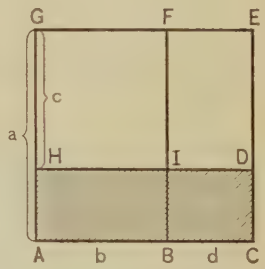
But $a \cdot b = GB$, $a \cdot d = FC$.

$\therefore ab + ad = GC$,

and $bc = HF$, $dc = FD$.

$\therefore ab + ad - bc - dc = AD$,

or $(a - c)(b + d) = ab + ad - bc - dc$.



Ex. Prove geometrically the following algebraic formulæ:[1]

(a) $a(b + c + d) = ab + ac + bd$.

(b) $(a + b)(c + d) = ac + bc + ad + bd$.

(c) $a(b - c) = ab - ac$.

(d) $(a + b)^2 = a^2 + 2ab + b^2$.

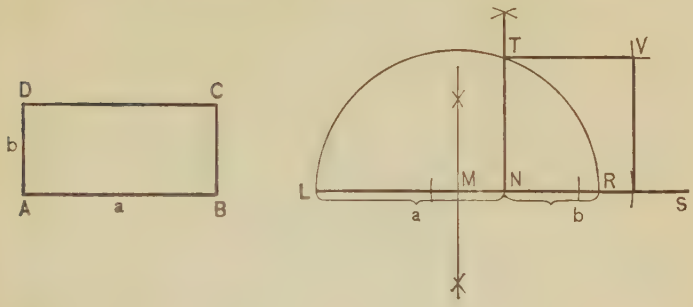
(e) $a^2 - b^2 = (a + b)(a - b)$.

(f) $(a - b)^2 = a^2 - 2ab + b^2$. HINT. $a^2 - ab + b^2 - ab$.

[1] The student is advised in the following exercises to shade the area which represents the left member.

PROPOSITION IX. PROBLEM

371. *To transform a rectangle into a square.*



Given rectangle $ABCD$.

Required a square $=$ $ABCD$.

Construction. On indefinite line LS, lay off LN and NR equal to a and b respectively. On $(a + b)$ as a diameter construct a semicircle. At N erect $\perp$ to the diameter meeting the semicircle at T. NT equals a side of the required square. Construct the square NV.

Proof. Area rectangle $= ab$.

Area square $= x^2$.

$\therefore\ x^2 = ab,\ x = \sqrt{ab}$.

Consequently, x is a mean proportional between a and b.

By the algebraic method construct:

Ex. **1.** A square equivalent to a parallelogram.

Ex. **2.** A square equivalent to twice a triangle.

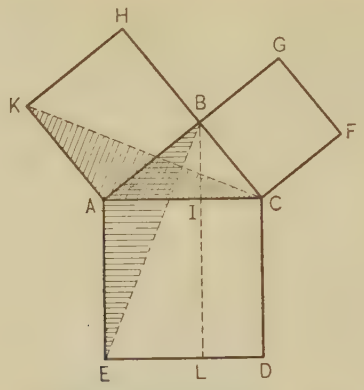
Ex. **3.** A square equivalent to one half a trapezoid.

Ex. **4.** A square equivalent to the sum of two triangles.

Ex. **5.** A square equivalent to the difference between a rectangle and a triangle.

Proposition X. Theorem

372. *In a right triangle the sum of the squares on the arms is equivalent to the square on the hypotenuse.*[1]



Given AH and BF squares on the arms, AD the square on the hypotenuse, of the right triangle ABC.

To prove sq. AD = sq. AH + sq. BF.

Proof

STATEMENTS	REASONS
1. Draw $BL \perp$ to ED.	1. Why?
2. $BL \parallel AE$.	2. Why?
3. Draw KC and BE.	3. Why?
4. $\begin{cases} BA = \text{an altitude of } \triangle KAC. \\ AI = \text{an altitude of } \triangle BAE. \end{cases}$	4. Why?
5. $KA = AB$, and $AC = AE$.	5. Hyp.
6. $\angle KAB = \angle EAC$.	6. Rt. $\measuredangle$ are $=$.
7. $\angle BAC = \angle BAC$.	7. Iden.
8. $\angle KAC = \angle BAE$.	8. Why?

[1] The shading in this figure is employed merely to point out the two congruent triangles.

9. $\triangle KAC \cong \triangle BAE$.	9. s. a. s. = s. a. s.
10. $\begin{cases} \triangle KAC = \frac{1}{2}(ABHK). \\ \triangle BAE = \frac{1}{2}(AELI). \end{cases}$	10. Why?
11. $\therefore$ Square AH = Rect. AL.	11. Why?
12. Square BF = Rect. DI.	12. Steps similar to (1 – 11).
13. $\therefore$ Sq. AH + Sq. BF = Sq. AD.	13. Why?

373. Cor. *The square on an arm of a right triangle is equivalent to the difference between the squares on the hypotenuse and the other arm.*

374. Note. The preceding theorem was first demonstrated by Pythagoras about 550 B.C., and is called after him the **Pythagorean Theorem**. The proof above was given by Euclid (about 300 B.C.).

Proposition XI. Problem

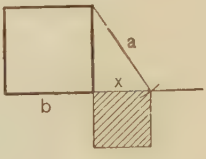
375. *To construct a square equivalent to the sum of two given squares.*

[The solution is left to the student. Apply Prop. X.]

Proposition XII. Problem

376. *To construct a square equivalent to the difference of two given squares.*

[The solution is left to the student.]



Ex. 1. To construct a square equivalent to the sum of three given squares.

Ex. 2. Construct x^2, given $x^2 = a^2 + b^2 - c^2$.

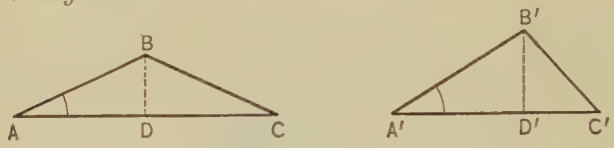
Ex. 3. Construct x^2, given $x^2 = 2\,a^2$.

Hint. $x^2 = a^2 + a^2$.

Ex. 4. Using above method, construct a square equivalent to three times a given square.

Proposition XIII. Theorem

377. *The areas of two triangles which have an angle of the one equal to an angle of the other are to each other as the products of the sides including the equal angles.*



Given $\triangle ABC$ and $A'B'C'$, $\angle A = \angle A'$.

To prove $$\frac{\triangle ABC}{\triangle A'B'C'} = \frac{AB \times AC}{A'B' \times A'C'}.$$

Proof.

1.	Draw the altitudes BD and $B'D'$.	**1.** Why?
2.	$\angle A = \angle A'$.	**2.** Hyp.
3.	$\therefore \triangle ABD \sim \triangle A'B'D'$.	**3.** (306).
4.	$\frac{BD}{B'D'} = \frac{AB}{A'B'}$	**4.** Why?
5.	$\frac{\triangle ABC}{\triangle A'B'C'} = \frac{AC \times BD}{A'C' \times B'D'}$	**5.** (360).
	$= \frac{AC}{A'C'} \times \frac{BD}{B'D'}$	
6.	$= \frac{AC}{A'C'} \times \frac{AB}{A'B'}$	**6.** Sub.
7.	$= \frac{AC \times AB}{A'C' \times A'B'}$.	**7.** Why?

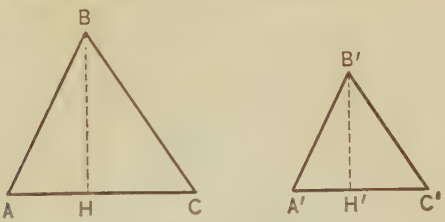
Ex. **1.** In triangles ABC and $A'B'C'$, $\angle A = \angle A'$, $AB = 6$ in., $AC = 9$ in., $A'B' = 1$ ft., and $A'C' = 2$ ft. Find the ratio of $\triangle ABC$ to $\triangle A'B'C'$.

Ex. **2.** Triangle ABC is equivalent to triangle $A'B'C'$, and $\angle A = \angle A'$. Find AC if $AB = 2$ in., $A'B' = 3$ in., and $A'C' = 4$ in.

Ex. **3.** If $\triangle ABC = \triangle A'B'C'$, and $\angle A = \angle A'$, then
$$AB : A'B' = A'C' : AC.$$

Proposition XIV. Theorem

378. *Similar triangles are to each other as the squares of their corresponding sides.*



Given $\triangle ABC \sim \triangle A'B'C'$.

To prove $\dfrac{\triangle ABC}{\triangle A'B'C'} = \dfrac{\overline{AB}^2}{\overline{A'B'}^2} = \dfrac{\overline{AC}^2}{\overline{A'C'}^2} = \dfrac{\overline{BC}^2}{\overline{B'C'}^2}$.

Proof

STATEMENTS	REASONS
1. Draw corresponding altitudes BH and $B'H'$.	1. Why?
2. $\angle A = \angle A'$.	2. Hyp.
3. Rt. $\triangle ABH \sim$ rt. $\triangle A'B'H'$.	3. (306).
4. $\dfrac{\triangle ABC}{\triangle A'B'C'} = \dfrac{AC \times BH}{A'C' \times B'H'} = \dfrac{AC}{A'C'} \times \dfrac{BH}{B'H'}$.	4. (360).
5. $\dfrac{BH}{B'H'} = \dfrac{AC}{A'C'}$.	5. (314).
6. $\therefore \dfrac{\triangle ABC}{\triangle A'B'C'} = \dfrac{AC}{A'C'} \times \dfrac{AC}{A'C'} = \dfrac{\overline{AC}^2}{\overline{A'C'}^2}$.	6. Sub.
7. In like manner $\dfrac{\triangle ABC}{\triangle A'B'C'} = \dfrac{\overline{BC}^2}{\overline{B'C'}^2} = \dfrac{\overline{AB}^2}{\overline{A'B'}^2}$.	7. Steps (1–6).

379. *Similar triangles are to each other as the squares of their corresponding altitudes.*

Ex. 1. Similar triangles are to each other as the squares of their perimeters.

Ex. 2. The sides of a triangle are 4, 7, and 8. What are the sides of a similar triangle whose area is four times as large?

Ex. 3. Find the ratio of two similar triangles, two of whose corresponding sides are 2 in. and 3 in. respectively.

Ex. 4. Compare the areas of two similar triangles if two corresponding sides have the ratio of 2 : 3.

Ex. 5. The areas of two similar triangles are to each other as 9 : 25. If the perimeter of the first is 36, what is the perimeter of the second?

Ex. 6. The area of one equilateral triangle is 100 sq. in. What is the area of a second equilateral triangle, if its base is twice the base of the first?

Ex. 7. A line is drawn connecting the mid-points of two sides of a triangle. What part of the given triangle is the triangle cut off by the given line?

Ex. 8. What is the ratio of the two right triangles CBD and BDA in the figure on page 186, Prop. XXX?

Ex. 9. If in Ex. 8, the two arms are 5 and 12, find the numerical ratio of the two right triangles in that exercise.

Ex. 10. The nonparallel sides of a trapezoid are prolonged to meet at a point. If the bases of the trapezoid are 20 and 12, what is the ratio of the areas of the triangles in the figure?

Ex. 11. Similar triangles are to each other as the squares of their corresponding (*a*) medians, (*b*) angle-bisectors.

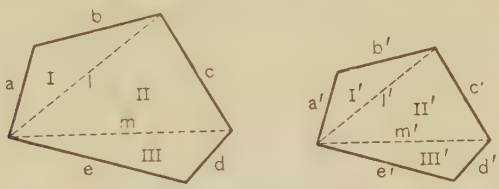
Ex. 12. Squares are constructed on the base and the altitude of an equilateral triangle. Compare their areas.

Ex. 13. The areas of two similar triangles are 64 and 225 respectively. The altitude of the first triangle is 6. What is the length of the corresponding altitude of the other?

Ex. 14. The bases of a trapezoid are 12 and 8 respectively. The nonparallel sides are prolonged until they meet. If the area of the trapezoid is 90, what is the area of the smaller triangle?

Proposition XV. Theorem

380. *Similar polygons are to each other as the squares of their corresponding sides.*



Given a and a' corresponding sides of two similar polygons whose areas are S and S' respectively.

To prove $$\frac{S}{S'} = \frac{a^2}{a'^2}.$$

Proof

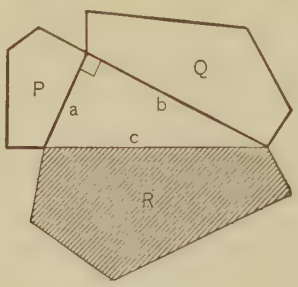
STATEMENTS	REASONS
1. From two corresponding vertices draw all possible diagonals.	1. Why?
2. $\triangle \mathrm{I} \sim \triangle \mathrm{I}'$, $\triangle \mathrm{II} \sim \triangle \mathrm{II}'$, etc.	2. (315).
3. $\dfrac{\triangle \mathrm{I}}{\triangle \mathrm{I}'} = \dfrac{a^2}{a'^2}$.	3. (378).
4. $\dfrac{\triangle \mathrm{I}}{\triangle \mathrm{I}'} = \left(\dfrac{l^2}{l'^2}\right) = \dfrac{\triangle \mathrm{II}}{\triangle \mathrm{II}'} = \left(\dfrac{m^2}{m'^2}\right) = \dfrac{\triangle \mathrm{III}}{\triangle \mathrm{III}'}$.	4. (378).
5. $\dfrac{\triangle \mathrm{I}}{\triangle \mathrm{I}'} = \dfrac{\triangle \mathrm{II}}{\triangle \mathrm{II}'} = \dfrac{\triangle \mathrm{III}}{\triangle \mathrm{III}'}$.	5. Why?
6. $\dfrac{\triangle \mathrm{I} + \triangle \mathrm{II} + \triangle \mathrm{III}}{\triangle \mathrm{I}' + \triangle \mathrm{II}' + \triangle \mathrm{III}'} = \dfrac{\triangle \mathrm{I}}{\triangle \mathrm{I}'}$.	6. (284).
7. $S : S' = \triangle \mathrm{I} : \triangle \mathrm{I}'$	7. Why?
8. $\therefore S : S' = a^2 : a'^2$.	8. Why?

381. Cor. 1. *Similar polygons are to each other as the squares of their corresponding diagonals.*

382. Cor. 2. *Similar polygons are to each other as the squares of their perimeters.*

Proposition XVI. Theorem

383. *If similar polygons are constructed on the three sides of a right triangle, the sum of the polygons on the arms is equivalent to the polygon on the hypotenuse.*



Given the similar polygons P, Q, and R constructed, respectively, on the arms a and b and the hypotenuse c of the rt. $\triangle abc$.

To prove $P + Q = R$.

Proof.

$$\frac{P}{R} = \frac{a^2}{c^2} \text{ and } \frac{Q}{R} = \frac{b^2}{c^2}. \quad (380)$$

$$\therefore \frac{P+Q}{R} = \frac{a^2+b^2}{c^2} = \frac{c^2}{c^2} = 1. \quad (\text{Ax. } 2, 329.)$$

i.e. $P + Q = R$. (Why?)

Ex. **1.** The areas of two similar polygons are respectively 3 sq. in. and 12 sq. in., and a side of the first is 6 in. Find the corresponding side of the second.

Ex. **2.** Two corresponding sides of two similar polygons are 4 in. and 3 in. respectively, the area of the greater polygon is 1 sq. ft. What is the area of the smaller polygon?

Ex. **3.** The area of a polygon is twice the area of a similar polygon. Find the ratio of any two corresponding sides.

Ex. **4.** On a map whose scale is 1 : 1000, how many square feet are represented by a polygon equivalent to 2 sq. in.?

Ex. 5. Corresponding sides of 2 similar polygons have the ratio of 5 to 9, the sum of their areas is 212 sq. ft. Find the area of each figure.

Ex. 6. Two similar parallelograms are to each other as the products of their diagonals.

Ex. 7. Construct a triangle equivalent to two thirds of a given triangle, and similar to it.

Ex. 8. Construct a triangle similar to a given triangle and three times as large.

Ex. 9. Divide a triangle into two equivalent parts by a line parallel to one of its sides.

Ex. 10. Divide a triangle into three equivalent parts by lines parallel to one of its sides.

Ex. 11. Construct a polygon equivalent to two thirds of a given polygon and similar to it.

Ex. 12. *Construct an equilateral triangle equivalent to the sum of two given equilateral triangles.*

Ex. 13. *Construct an equilateral triangle equivalent to the difference of two given equilateral triangles.*

Ex. 14. Construct an isosceles right triangle equivalent to the sum of two given isosceles right triangles.

Ex. 15. Construct an isosceles right triangle equivalent to the difference of two isosceles right triangles.

Ex. 16. *Construct a triangle similar to two given similar triangles and equivalent to their sum.*

Ex. 17. Construct a polygon similar to two given similar polygons and equivalent to their sum.

Ex. 18. Construct a polygon similar to two given similar polygons and equivalent to their difference.

Ex. 19. Construct an equilateral triangle equivalent to the sum of three given equilateral triangles.

Ex. 20. Two corresponding sides of two similar polygons are 12 feet and 5 feet respectively. Find the corresponding side of a similar polygon equivalent to their sum.

MISCELLANEOUS EXERCISES

THEOREMS

Ex. 1. If E is any point in the diagonal AC of the parallelogram $ABCD$, prove that $\triangle AEB = \triangle ADE$.

Ex. 2. A straight line passing through the intersections of the diagonals of a parallelogram divides the figure into two equivalent parts.

Ex. 3. *The area of a circumscribed polygon is equivalent to one half the product of perimeter and radius.*

Ex. 4. If through any point in a diagonal of a parallelogram parallels are drawn to the sides, four parallelograms are formed, of which the two which do not contain the diagonal are equivalent.

Ex. 5. If any point within a parallelogram be joined to the four vertices, the sum of either pair of opposite triangles is equivalent to one half the parallelogram.

Ex. 6. The lines joining the mid-points of the sides of a quadrilateral in succession form a parallelogram equivalent to one half the quadrilateral.

Ex. 7. A rhombus and a square have equal perimeters. Compare their areas, given one angle of the rhombus is 60°.

Ex. 8. If on the line joining the mid-points of two sides of a triangle a parallelogram is constructed having two of its vertices in the base of the triangle, the parallelogram is equivalent to one half the triangle.

Ex. 9. The nonparallel sides of a trapezoid form with the diagonals two equivalent triangles.

Ex. 10. If from the point of intersection of the medians of any triangle lines are drawn to the three vertices, they form with the sides three equivalent triangles.

Ex. 11. If in the triangle ABC, D and F are the mid-points of the sides AB and AC respectively, the $\triangle ADC = \triangle ABF$.

Ex. 12. *The area of an equilateral triangle whose side equals a is* $\frac{a^2}{4}\sqrt{3}$.

Ex. 13. *The area of a rhombus is equal to half the product of the diagonals.*

PROBLEMS OF COMPUTATION

Ex. 1. The side of an equilateral triangle is 10 in. Find the area.

Ex. 2. Find the area of an isosceles triangle if the base is 6 and an arm 5.

Ex. 3. Find the area of a trapezoid whose bases are 9 and 11 respectively, and whose altitude is 12 ft.

Ex. 4. Find the area of a rhombus whose diagonals are 9 and 10 ft. respectively.

Ex. 5. Find the area of quadrilateral $ABCD$ if $AB = 10$, $BC = 24$, $CD = 30$, $AD = 28$, diagonal $AC = 26$.

Ex. 6. A side of equilateral triangle ABC is 8. Find the side of an equilateral triangle equivalent to three times triangle ABC.

Ex. 7. The perimeter of a rectangle is 20 m., one side is 6 m. Find the area.

Ex. 8. What is the side of a square whose area is 900 sq. m.? n sq. m.?

Ex. 9. The area of a rhombus is equal to m, and one diagonal is equal to d. Find the other diagonal.

Ex. 10. The area of a trapezoid is 400 sq. m., its altitude is 8 m. Find the length of the line joining the mid-points of the nonparallel sides.

Ex. 11. The hypotenuse of a right triangle is 20, and the projection of one arm upon the hypotenuse is 4. What is its area?

Ex. 12. A farmer wishes to determine the area of a pentagonal field. He measures the lines $AB = 4$ rods, $BC = 13$ rods, $CD = 14$ rods, $DE = 5$ rods, $EA = 12$ rods, $AC = 15$ rods, and $AD = 13$ rods. How many square rods does the field contain?

Ex. 13. The base and altitude of a triangle are 12 and 20 respectively. At a distance of 6 from the base, a parallel is drawn to the base. Find the areas of the two parts of the triangle.

Ex. 14. Find the area of a rectangle having one side equal to 6 and a diagonal equal to 10.

Ex. 15. A polygon with perimeter of 20 ft. is circumscribed about a circle whose radius is 3 ft. Find the area of the polygon.

Ex. 16. Find the area of a triangle whose base is 10 inches and whose base angles are 120° and 30° respectively.

Ex. 17. Find the side of an equilateral triangle equivalent to a rectangle, whose base and altitude are 10 and 15 respectively.

Ex. 18. The chord of an arc is 42 in.; the chord of one half that arc is 29 in. Find the diameter of the circle.

Ex. 19. The base of a triangle is 15 ft., its area is 60 sq. ft. Find the area of a similar triangle whose corresponding altitude is 6 ft.

Ex. 20. An arm of an isosceles trapezoid is 13 in. and its projection on the longer base is 5 in.; the longer base is 17 in. Find the area of the trapezoid.

Ex. 21. The arms of two isosceles right triangles are 3 and 2 respectively. Find the arm of an isosceles right triangle equivalent to their difference.

Ex. 22. The sides of a triangle are as 8:15:17. Find the sides if the area is 960 sq. ft.

Ex. 23. The sides of a triangle are 8, 15, and 17. Find the radius of the inscribed circle.

Hint. Express in the form of an equation the fact that the area of the triangle is equal to the sum of the three triangles whose vertex is the incenter, and whose bases are the sides of the triangle.

Ex. 24. Find the area of an equilateral polygon of 12 sides inscribed in a circle whose radius is 4 in.

Ex. 25. The sides of a triangle are 6, 7, and 8 ft. Find the areas of the two parts into which the triangle is divided by the bisector of the angle included by 6 and 7.

Ex. 26. Find the area of an equilateral triangle whose altitude is equal to h.

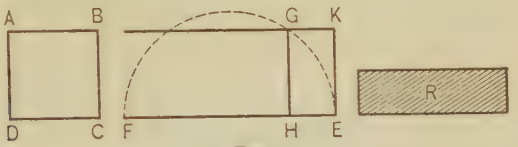
Ex. 27. Find the area of a rhombus if the sum of its diagonals is 24 feet and their ratio is as 3 : 5.

Ex. 28. Upon the diagonal of a rectangle 16 ft. by 12 ft., a triangle equivalent to the rectangle is constructed. What is its altitude?

PROBLEMS OF CONSTRUCTION

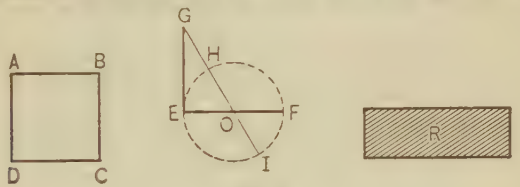
Ex. 1. To construct a rectangle equivalent to a given square, having the sum of its base and altitude equal to a given line.

Hint. If FE is the given line and $ABCD$ the given square, make $EK = AB$.



Ex. 1

Ex. 2. To construct a rectangle equivalent to a given square, having the difference of its base and altitude equal to a given line.



Ex. 2

Hint. If $ABCD$ is the given square and EF the given line, make $EG = AB$.

Ex. 3. To transform a rectangle into another one, having a given side.

Ex. 4. To transform a square into an isosceles triangle, having a given base.

Ex. 5. To transform a rectangle into a parallelogram having a given diagonal.

Ex. 6. To divide a triangle into three equivalent parts by lines parallel to a median.

Ex. 7. To bisect a parallelogram by a line perpendicular to a side.

Ex. 8. To divide a parallelogram into three equivalent parts by lines drawn through a vertex.

Ex. 9. To bisect a trapezoid by a line drawn through a vertex.

Ex. 10. Divide a pentagon into four equal parts by lines drawn through one of its vertices.

MISCELLANEOUS

Ex. 1. Show how to construct a line making an angle of 60° with a given line and tangent to a given circle.

Ex. 2. One side of a rhombus is 20; the longer diagonal is 32. Find the other diagonal and the area.

Ex. 3. Prove that the bisectors of the angles of a rectangle inclose a square.

Ex. 4. Through a given point outside a given line to draw two lines, each making an angle of 45° with the given line.

Ex. 5. A tower casts a shadow 120 feet long when a vertical 15-foot pole casts a shadow 10 feet long. How high is the tower?

Ex. 6. Find the locus of the centers of all circles tangent to two concentric circles.

Ex. 7. How far is a man from his starting point, if he first travels 5 miles north, then 12 miles east, and then 4 miles south?

Ex. 8. Prove that the bisector of the right angle in any right triangle, if produced, passes through the center of the square having the hypotenuse as a side.

Ex. 9. Construct a square equivalent to a given hexagon.

Ex. 10. Two equilateral triangles have a side of one equal to the altitude of the other. Find the ratio of their areas.

Ex. 11. Find the locus of the mid-points of all parallel chords in a circle.

Ex. 12. On the base of a given triangle to construct an isosceles triangle equal in area to the given triangle.

Ex. 13. Prove that the area of a triangle having an angle of 30° is equal to one fourth the product of the two sides including the angle of 30°.

Ex. 14. Through a given point construct a line equidistant from two given points (2 sol.).

Ex. 15. Find the side of an equilateral triangle whose area is $25\sqrt{3}$.

Ex. 16. Divide a given arc of a circle into two parts which shall have their chords in a given ratio.

Ex. **17**. Two chords intersect within a circle; the segments of one are each 4, the total length of the other is 10. Find the length of its segments.

Ex. **18**. A house is 40 feet long, 30 feet wide, and 25 feet high to the roof and 35 feet to the ridgepole. Find the area of its exterior surface.

Ex. **19**. To construct a square which shall have a given ratio to a given square.

Ex. **20**. Prove: If the two diagonals of a quadrilateral divide each other in the same ratio, the quadrilateral is a trapezoid.

Ex. **21**. The radius of a circle is 13 inches. Through a point 5 inches from the center any chord is drawn. What is the product of the two segments of the chord?

Ex. **22**. The altitude upon the base of a triangle is 12. The two segments of the base made by the altitude are 9 and 16 respectively. Show that the triangle is a right triangle.

Ex. **23**. A side of a rhombus is 13 meters and one of its diagonals is 24 meters. Find its area.

Ex. **24**. AB is any chord. AC is a tangent and CDE is a secant parallel to AB and cutting the circle in D and E. Prove $AC : AE = DC : BE$.

Ex. **25**. Divide a given line so that the product of its two segments equals a given square.

Ex. **26**. On one side of a triangle construct a rhombus equivalent to the given triangle.

BOOK V

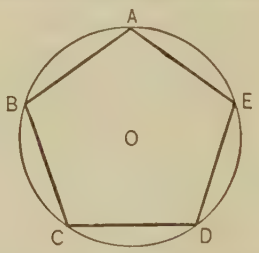
REGULAR POLYGONS. MEASUREMENT OF THE CIRCLE

REGULAR POLYGONS

384. Def. A **regular polygon** is a polygon which is both equiangular and equilateral.

Proposition I. Theorem

385. *An equilateral polygon inscribed in a circle is regular.*



Given circle O and inscribed equilateral polygon $ABCDE$.

To prove polygon $ABCDE$ is regular.

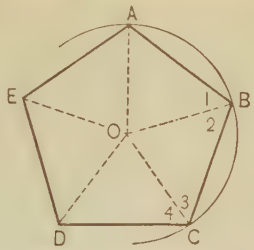
Proof

STATEMENTS	REASONS
1. $AB = BC = CD = DE \cdots$.	1. Hyp.
2. $\overset{\frown}{AB} = \overset{\frown}{BC} = \overset{\frown}{CD} = \overset{\frown}{DE} \cdots$.	2. (187).
3. $\overset{\frown}{ABC} = \overset{\frown}{BCD} = \overset{\frown}{CDE} = \overset{\frown}{DEA} \cdots$	3. Ax. 2.
4. $\therefore \angle B = \angle C = \angle D = \angle E = \angle A$.	4. Why?
5. The polygon is regular.	5. (384).

[Practical applications, pages 297–300.]

Proposition II. Theorem

386. *A circle can be circumscribed about any regular polygon.*



Given the regular polygon *ABCDE*.

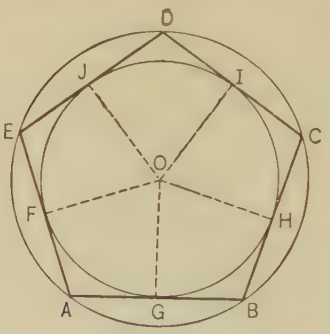
To prove that a circle can be circumscribed about *ABCDE*.

Proof

STATEMENTS	REASONS
1. Construct a circle through A, B, and C and let O be its center.	1. To pass a circle through three points not in a same st. line.
2. Draw OA, OB, OC, OD.	2. Why?
3. $\angle ABC = \angle BCD$.	3. Hyp.
4. $OB = OC$.	4. Why?
5. $\angle 2 = \angle 3$.	5. (71).
6. $\angle 1 = \angle 4$.	6. Why?
7. $AB = CD$.	7. Hyp.
8. $\triangle OAB \cong \triangle OCD$.	8. s. a. s. = s. a. s.
9. $OA = OD$.	9. Why?
10. The $\odot$ passes through D.	10. Why?
11. In like manner it can be proved that the circle passes through the other vertices of the polygon.	11. Steps (1–10).
12. A circle may be circumscribed about the given polygon.	12. Why?

Proposition III. Theorem

387. *A circle can be inscribed in any regular polygon.*



Given regular polygon *ABCDE*.

To prove a circle can be inscribed in the polygon.

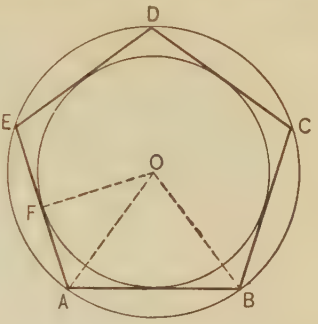
Proof

STATEMENTS	REASONS
1. Let a circle be circumscribed about the given regular polygon.	**1.** (386).
2. From center *O* of circumscribed circle draw perpendiculars to each of the sides.	**2.** Why?
3. $AB = BC = CD = DE = EA$.	**3.** Hyp.
4. $OG = OH = OF$, etc.	**4.** (197).
5. With *O* as center and with *OG* as a radius describe a circle.	**5.** Why?
6. The circle passes through points *F*, *G*, *H*, *I*, *J*.	**6.** Why?
7. Sides *AB*, *BC*, *CD*, etc. are tangents to the circle having *OG* as radius.	**7.** (203).
8. Hence a circle has been inscribed in the given regular polygon.	**8.** Each side of the polygon is a tangent to the ⊙.

388. DEF. The **center** (O) **of a regular polygon** is the common center of the circumscribed and inscribed circles of the polygon.

389. DEF. The **radius of a regular polygon** is the radius of the circumscribed circle, as OA.

390. DEF. The **central angle of a regular polygon** is the angle between two radii drawn to the ends of one side, as AOB.



391. DEF. The **apothem of a regular polygon** is the radius of the inscribed circle, as OF.

392. COR. *The central angle of a regular polygon of n sides is equal to $\frac{4}{n}$ right angles.*

Ex. 1. If $ABCDEF$ is a regular hexagon, then

$$\angle ADC = \angle AEC = \angle AFC.$$

Ex. 2. If two diagonals AC and BE of a regular polygon $ABCDEFG$ intersect in Q, then $AQ \times QC = BQ \times QE$.

Ex. 3. If in the regular heptagon $ABCDEFG$, the prolongations of AB and the diagonal of EC meet in H, then $HB \times HA = HC \times HE$.

Ex. 4. *Any central angle of a regular polygon is the supplement of an angle of the polygon.*

Ex. 5. A triangle is regular if its circumscribed and inscribed circles are concentric.

Ex. 6. A polygon is regular if its circumscribed and inscribed circles are concentric.

Ex. 7. How many degrees are there in a central angle of a regular triangle? In a regular octagon? In a regular hexagon?

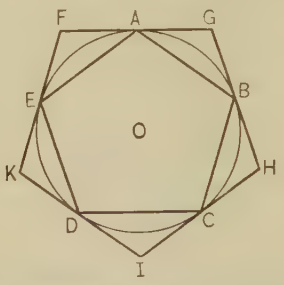
Ex. 8. Find the apothem of a regular hexagon having each of its sides 6 inches.

Proposition IV. Theorem

393. *If a circle is divided into any number of equal parts:*

(1) *The chords joining the successive points of division form a regular inscribed polygon;*

(2) *Tangents drawn at the points of division form a regular circumscribed polygon.*



Given the circle ACE divided into the equal arcs AB, BC, CD, etc., and chords AB, BC, CD, etc., and tangents FG, GH, HI, etc., at points of division.

(1) **To prove** $ABCDE$ is a regular polygon.

Proof.

1.	$\overset{\frown}{AB} = \overset{\frown}{BC} = \overset{\frown}{CD}$, etc.	(Why?)
2.	$AB = BC = CD$, $\cdots$	(Why?)
3.	$ABCDE$ is a regular polygon	(385)

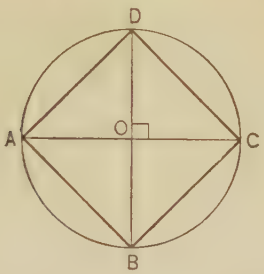
(2) **To prove** $FGHIK$ is a regular polygon.

Proof.

1.	$\angle GAB = \angle GBA = \angle CBH = \angle HCB$, etc.	(Why?)
2.	$AB = BC = CD$, etc.	(Why?)
3.	$\triangle ABG$, CBH, $\cdots$ are isos. and $\cong$	(Why?)
4.	$\angle G = \angle H = \angle I$, etc.	(Why?)
5.	$AG = GB = BH = HC$, etc.	(Why?)
6.	$GH = HI = IK$, etc.	(Ax. 2.)
7.	$\therefore$ $FGHIK$ is a regular polygon.	(384)

Proposition V. Problem

394. *To inscribe a square in a given circle.*



Given circle O.

Required to inscribe a square in the circle.

Construction. Draw any diameter AC. Construct the diameter $DB \perp AC$. Join A, B, C, D in order. $ABCD$ is the required square.

Proof

STATEMENTS	REASONS
1. $DB \perp AC$.	1. Why?
2. $AD = DC = CB = BA$.	2. (125).
3. $ABCD$ is a regular quadrilateral; *i.e.* a square.	3. (385).

395. Cor. 1. *By bisecting the central angles, the arcs AB, BC, etc., will be bisected, and a regular octagon can be inscribed in the circle. By repeating the process, regular polygons of* 16, 32, ···, *sides can be constructed.*

396. Cor. 2. *If R is the radius of a circle, the side of the inscribed square equals* $R\sqrt{2}$.

Ex. 1. Circumscribe a square about a circle.

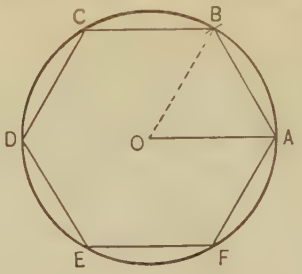
Ex. 2. Circumscribe a regular octagon about a given circle.

Ex. 3. Construct a regular octagon having a given side.

Ex. 4. Find the area of a square, if its radius is equal to r

Proposition VI. Problem

397. *To inscribe a regular hexagon in a given circle.*



Given circle O.

Required to inscribe a regular hexagon in the circle.

Construction. In the given circle ACD, draw the radius AO. From A as a center, with a radius equal to OA, draw an arc meeting the circle in B. Draw AB. Then arc AB will go exactly six times in the circle. Chord AB goes six times also as a chord.

Proof

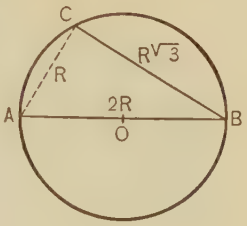
STATEMENTS	REASONS
1. Draw OB.	1. Why?
2. $OA = OB = AB$.	2. Why?
3. $\angle O = \angle 60°$.	3. Why?
4. $\overset{\frown}{AB} = \overset{\frown}{60°}$.	4. Why?
5. $\overset{\frown}{AB}$ goes exactly 6 times in the $\odot$.	5. Why?
6. $\therefore AB = BC = CD$, etc.	6. (187).
7. $ABCDEF$ is a regular inscribed hexagon.	7. (385).

398. Cor. 1. *The side of a regular inscribed hexagon is equal to the radius of the circle.*

399. Cor. 2. *By joining the alternate vertices of an inscribed regular hexagon, an inscribed equilateral triangle is formed.*

400. Cor. 3. *Regular polygons of 3, 6, 12, 24, etc., sides can be inscribed in and circumscribed about a given circle.*

401. Cor. 4. *If R is the radius of a circle, the side of the inscribed equilateral triangle is $R\sqrt{3}$.*



Hint. If AB is a diameter and $AC = R$, then BC is the required side.

Ex. 1. *If the radius of a circle is R, the apothem of the inscribed hexagon is equal to $\frac{R}{2}\sqrt{3}$.*

Ex. 2. Circumscribe a regular hexagon about a given circle.

Ex. 3. Construct a regular polygon of twelve sides, having given a side.

Ex. 4. Find the area of a regular hexagon if its radius is equal to R.

Ex. 5. The apothem of an equilateral triangle is equal to one half its radius.

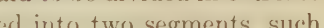
Ex. 6. The area of an inscribed equilateral triangle is equal to one half the area of the inscribed regular hexagon.

* Ex. 7. The areas of triangles inscribed in equal circles are to each other as the products of their three sides. (342)

Ex. 8. A square constructed on a diameter of a circle is equivalent to twice the area of the inscribed square.

402. Def. A straight line is said to be divided in **extreme and mean ratio** when it is divided into two segments, such that *the greater is the mean proportional between the smaller and the whole line.*

Thus, AB is divided by C in extreme and mean ratio, if



$$AB : AC = AC : CB.$$

Proposition VII. Problem

403. *To divide a line in extreme and mean ratio.*

Given line $AB = a$.

Required to divide a in extreme and mean ratio.

Analysis. Suppose AC, or x, is the greater of the required segments, then $CB = a - x$.

Hence, $a : x = x : a - x$.

$\therefore x^2 = a^2 - ax$.

Transposing, $x^2 + ax = a^2$.

Completing the square, $x^2 + ax + \left(\frac{a}{2}\right)^2 = a^2 + \left(\frac{a}{2}\right)^2$.

Extracting the root, $x + \frac{a}{2} = \sqrt{a^2 + \left(\frac{a}{2}\right)^2}$.

$$\therefore x = \sqrt{a^2 + \left(\frac{a}{2}\right)^2} - \frac{a}{2}.$$

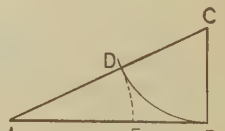
But since $\sqrt{a^2 + \left(\frac{a}{2}\right)^2}$ is the hypotenuse of a right triangle whose arms are a and $\frac{a}{2}$, x is easily constructed.

Construction. At B draw $CB = \frac{a}{2}$ and $\perp AB$.

$\left[\text{Then } AC = \sqrt{a^2 + \left(\frac{a}{2}\right)^2}.\right]$

On CA lay off $CD = CB \left(= \frac{a}{2}\right)$.

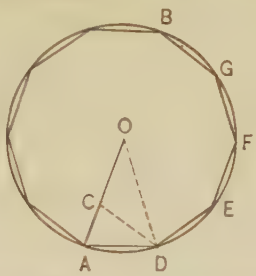
$\left[\text{Then } AD = \sqrt{a^2 + \left(\frac{a}{2}\right)^2} - \frac{a}{2} \text{ or } x.\right]$



On AB lay off $AF = AD$. Then AB is divided as required.

Proposition VIII. Problem

404. *To construct the side of a regular decagon inscribed in a given circle.*



Given $\odot O$.

Required to construct the side of a regular inscribed decagon.

Construction. Divide a radius OA in extreme and mean ratio. Draw chord AD equal to OC, the greater segment of OA. AD is a side of the required decagon.

Proof. Draw DO and DC.

$$AO : OC = OC : CA, \qquad \text{(Con.)}$$

or since $$OC = AD,$$

$$AO : AD = AD : AC. \qquad \text{(Sub.)}$$

$$\therefore \triangle OAD \sim \triangle CAD. \qquad (311)$$

$$\therefore \angle O = \angle CDA. \qquad \text{(Why?)}$$

But since $\triangle OAD$ is isosceles, the similar triangle DAC must be isosceles,

or, $$CD = AD = CO,$$

whence $$\angle O = \angle CDO. \qquad \text{(Why?)}$$

But $$\angle O = \angle CDA.$$

$$\therefore 2\angle O = \angle ADO = \angle A. \qquad \text{(Ax. 2.)}$$

But $$\angle O + \angle D + \angle A = 2 \text{ rt. } \measuredangle.$$

$$\therefore 5\angle O = 2 \text{ rt. } \measuredangle,$$

or $$\angle O = \tfrac{2}{5} \text{ rt. } \angle, \text{ or } \tfrac{1}{10} \text{ of 4 rt. } \measuredangle.$$

$\therefore \overset{\frown}{AD}$ is $\frac{1}{10}$ of the circle, and AD is a side of an inscribed regular decagon.

405. Cor. 1. *By joining the alternate vertices A, E, G, etc., a regular inscribed pentagon is formed.*

406. Cor. 2. *Regular polygons of 5, 10, 20, etc., sides can be inscribed in and circumscribed about a given circle.*

Ex. **1.** If R is the radius of a circle, the side of the inscribed regular decagon equals $\frac{R}{2}(\sqrt{5}-1)$.

Hint. Simplify the value of x found in the analysis of Prop. VII. ($a = R$.)

Ex. **2.** Construct an angle of 36°.

Ex. **3.** Construct a regular decagon, having given a side.

Ex. **4.** Divide a right angle into five equal parts.

Ex. **5.** *The diagonals of a regular pentagon are equal.*

Ex. **6.** The diagonals of a regular pentagon divide each other in extreme and mean ratio.

Ex. **7.** The area of a regular inscribed hexagon is a mean proportional between the areas of the inscribed and circumscribed equilateral triangles.

Ex. **8.** *Any radius of a regular polygon bisects an angle of the polygon.*

Ex. **9.** The diagonals drawn from the vertex of a regular decagon divide that angle into eight equal parts.

Ex. **10.** Construct a regular pentagon, having given a side.

Ex. **11.** Explain a method by which we could construct an angle of 3°.

Ex. **12.** Prove: The square on a side of a regular inscribed pentagon equals the square of a radius and the square of a side of the regular inscribed decagon.

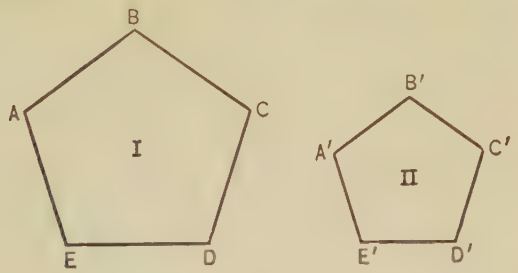
Ex. **13.** Find the side of a regular inscribed pentagon in terms of the radius (R) of its circumscribed circle.

Ex. **14.** Find the area of a regular pentagon if one side is equal to 4 inches.

Ex. **15.** Find the area of a regular decagon having side 4″.

PROPOSITION IX. THEOREM

407. *Regular polygons of the same number of sides are similar.*



Given I and II, two regular polygons, each of n sides.

To prove $\text{I} \sim \text{II}$.

Proof

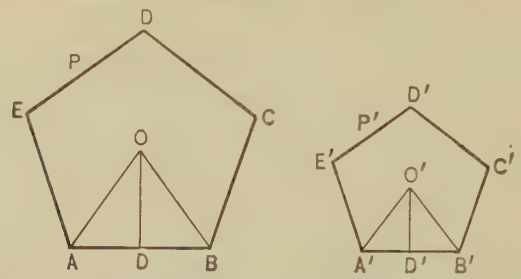
STATEMENTS	REASONS
1. Each $\angle$ of I $= \frac{n-2}{n}$ st. $\angle$. Each $\angle$ of II $= \frac{n-2}{n}$ st. $\angle$.	1. (157).
2. Each $\angle$ of I = each $\angle$ of II.	2. Why?
3. I and II are mutually equianglar.	3. Why?
4. $AB = BC = CD = DE = EA$. $A'B' = B'C' = C'D' = D'E' = E'A'$.	4. (384).
5. $\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'} = \frac{DE}{D'E'} = \frac{EA}{E'A'}$.	5. Why?
6. $\therefore$ I $\sim$ II.	6. (303).

Ex. 1. Construct a regular hexagon equivalent to one half a given regular hexagon.

Ex. 2. Construct a regular pentagon equivalent to the sum of two given regular pentagons.

Proposition X. Theorem

408. *The perimeters of regular polygons of the same number of sides have the same ratio as their radii, or as their apothems.*



Given P and P', the perimeters of two regular polygons of the same number of sides, having the radii OA and $O'A'$, and the apothems OD and $O'D'$ respectively.

To prove $P : P' = OA : O'A' = OD : O'D'$.

Proof

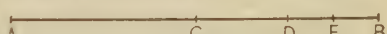
	STATEMENTS	REASONS
1.	$ABCDE \sim A'B'C'D'E'$.	1. (407).
2.	$P : P' = AB : A'B'$.	2. (318).
3.	$\angle AOB = \angle A'O'B'$.	3. (392).
4.	$AO = OB,\ A'O' = O'B'$.	4. Why?
5.	$\frac{AO}{A'O'} = \frac{OB}{O'B'}$.	5. Why?
6.	$\triangle AOB \sim \triangle A'O'B'$.	6. (311).
7.	$\frac{AB}{A'B'} = \frac{OA}{O'A'} = \frac{OD}{O'D'}$.	7. (303, 314).
8.	$\therefore P : P' = OA : O'A' = OD : O'D'$.	8. Why?

409. Cor. *The areas of regular polygons of the same number of sides are to each other as the squares of their radii, or as the squares of their apothems.*

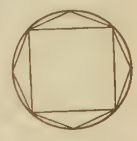
410. Def. A **constant** is a quantity that maintains the same value throughout a given discussion. A **variable** is a quantity whose value changes during the same discussion.

411. Def. If a variable x approaches a constant l so that the difference between l and x becomes and remains less than any assignable quantity different from zero, then l is said to be a **limit** of x.

1. For example; suppose a point p to take the positions C, D, E, etc., in succession, where $AC = \frac{1}{2}AB$, $CD = \frac{1}{2}CB$, $DE = \frac{1}{2}DB$, and so on, then the segments AC, AD, AE approach AB as a limit.



2. Inscribe a square in a circle, then inscribe an octagon by joining the mid-points of the four equal arcs to the vertices, and by continuing this process obtain regular inscribed polygons of 16, 32, 64, etc. sides. Obviously the area of the circle C is larger than the area of any of these polygons A, but the difference between these two areas, $C-A$, becomes less if we increase the number of sides of the polygon. By continually increasing the number of sides of the polygon, the area of the polygon will approach the area of the circle more and more, or $C-A$ can be made less than any assignable quantity, however small.



The area of the circle is defined as the limit of the area of the polygon.

3. Similarly the perimeter of an inscribed polygon increases with the number of sides and approaches a limit. We define this limit to be the length of the circle (*i.e.* the circumference).

4. The sequence of fractions .9, .99, .999, ···, .9999,···, approaches a limit equal to 1, since we can, by going along far enough in the set of numbers, find that all that follow it, differ from 1 by as small a number as we please to choose.

412. The symbol for "approaches a limit" is represented by a horizontal arrow, $\longrightarrow$. "x approaches l as a limit" is represented by $x \longrightarrow l$.

Note. In elementary geometry the variables that approach a limit as a rule cannot become equal to this limit. There are, however, cases in which variables become equal to their limits.

Ex. 1. What is the limit of the fractions represented by $\frac{1}{n}$, if n increases indefinitely?

Ex. 2. What is the limit of the sum $2 + 1 + \frac{1}{2} + \frac{1}{4} + \cdots$?

Ex. 3. What is the limit of the areas of regular circumscribed polygons of n sides if n assumes the values 4, 8, 16, $\cdots$ to infinity?

Ex. 3

Ex. 4. What is the limit of the perimeters of regular circumscribed polygons of n sides, if n assumes the values 4, 8, 16, $\cdots$ to infinity?

Ex. 5. If the number of sides of regular inscribed polygons is increased indefinitely, what is the limit of (*a*) the apothems? (*b*) the sides?

Ex. 6. If the number of sides of regular circumscribed polygons is increased indefinitely, what is the limit of the radii?

413. THEOREM. **If two variables, x and y, are always equal and x approaches a as a limit, then y approaches a as a limit.**

Since $a - x = a - y$, and $a - x$ can be made less than any conceivable number,

$a - y$ can be made less than any conceivable number.

$\therefore y \longrightarrow a$.

414. COR. **If two variables are always equal, and each approaches a limit, their limits are equal.** (PRINCIPLE OF LIMITS.)

415. The following theorems are true:

(A) If $x \longrightarrow a$, then $nx \longrightarrow na$.

(B) If $x \longrightarrow a$, then $\frac{1}{n}x \longrightarrow \frac{1}{n}a$.

(C) If $x \longrightarrow a$, and $y \longrightarrow b$, then $(x \pm y) \longrightarrow (a \pm b)$.

(D) If $x \longrightarrow a$, and $y \longrightarrow b$, then $\frac{x}{y} \longrightarrow \frac{a}{b}$. $[b \neq 0]$.

(E) If $x \longrightarrow a$, and $y \longrightarrow b$, then $xy \longrightarrow ab$.

Proof of (A): $(a - x) \longrightarrow 0$, $n(a - x) \longrightarrow 0$, $(na - nx) \longrightarrow 0$. $\therefore nx \longrightarrow na$.

416. DEF. The **length of a circle** (or, briefly, a circumference) is the limit of the perimeters of regular inscribed polygons, the number of whose sides is increased indefinitely.

Having defined a circumference, the meaning of the length of an arc is determined by § 228.

NOTE. It can be demonstrated that the limit of the perimeter of regular *circumscribed* polygons is the same as that of the inscribed ones. Hence a circumference can be defined as a common limit of the perimeters of regular inscribed and circumscribed polygons. For the purposes of this textbook, however, it is sufficient to consider the regular inscribed polygons only.

417. Teachers may desire to use the following *assumption* in place of the theory of limits. *Any property of a regular polygon which does not depend upon the number of sides of the polygon, is also a property of circles.*

418. DEF. A **sector** of a circle is a figure formed by an arc and the two radii drawn to the ends of the arc.

419. DEF. **Similar arcs, segments,** and **sectors** are those which correspond to equal central angles.

420. DEF. The **area of a circle** is the limit of the area of regular circumscribed polygons whose sides are increased indefinitely.

NOTE. It can be proved that the areas of regular inscribed polygons approach the same limit as the areas of the circumscribed polygons. For the purposes of this book, however, it is more convenient to consider the circumscribed polygon only.

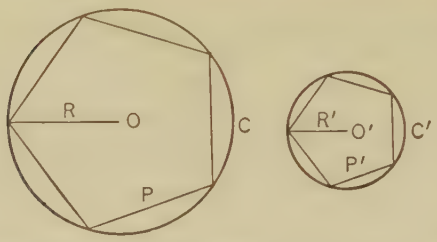
Ex. 1. Prove theorems (B), (C), (D), and (E) in section 415.

Ex. 2. Prove the lengths of similar arcs are in the same ratio as their radii.

Ex. 3. Show that the areas of similar segments are in the same ratio as the squares of their radii.

Proposition XI. Theorem

421. *Circumferences are to each other as their radii.*



Given circles O and O' in which C, C' and R, R' represent their circumferences and radii respectively.

To prove $C : C' = R : R'$.

Proof

STATEMENTS	REASONS
1. In each circle inscribe regular polygons of n sides having perimeters P and P'.	1. Why?
2. $P : P' = R : R'$.	2. (408).
3. $P : R = P' : R'$.	3. (279).
4. $P \longrightarrow C$ and $P' \longrightarrow C'$.	4. (416).
5. $\frac{P}{R} \longrightarrow \frac{C}{R}$, $\frac{P'}{R'} \longrightarrow \frac{C'}{R'}$.	5. (415 B).
6. $\therefore \frac{C}{R} = \frac{C'}{R'}$.	6. (414).
7. $\frac{C}{C'} = \frac{R}{R'}$, *i.e.* $C : C' = R : R'$.	7. (279).

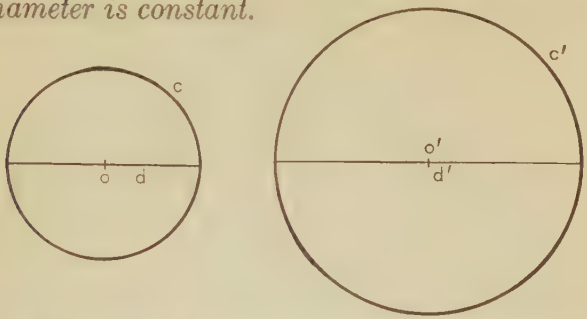
Alternate Proof

1. Statement (1) above.	1. Why?
2. $P : P' = R : R'$.	2. (408).
3. $\therefore C : C' = R : R'$.	3. (417).

422. *Circumferences are to each other as their diameters.*

PROPOSITION XII. THEOREM

423. *The ratio of the circumference of any circle to its diameter is constant.*



Given any circle O with circumference C and diameter d.

To prove that $C \div d$ = a constant.

Proof

STATEMENTS	REASONS
1. Construct a fixed circle O' with circum. $= C'$ and diameter $= d'$.	1. Why?
2. $C : C' = d : d'$.	2. (421).
3. $C : d = C' : d'$.	3. (279).
4. But $\frac{C'}{d'}$ = a constant.	4. Circle O' is fixed.
5. $\therefore \frac{C}{d}$ = a constant.	5. Sub.

424. The constant $\frac{C}{d}$ is represented by the Greek letter π.

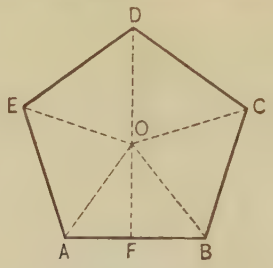
Approximate values of π are $3\frac{1}{7}$, 3.14, 3.1416.

Since $\frac{C}{d} = \pi$, then $C = \pi d = 2\pi R$.

425. *An arc of $m°$ in a circle of radius R has its length equal to* $\frac{m}{360} \times 2\pi R$.

Proposition XIII. Theorem

426. *The area of a regular polygon is equal to one half the product of its perimeter and apothem.*



Given S the area, P the perimeter, and r the apothem of the polygon $ABCDE$, having n sides.

To prove $S = \frac{1}{2} P \times r$.

Proof

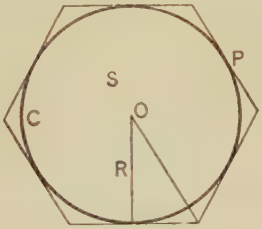
STATEMENTS	REASONS
1. Draw all the radii of the circumscribing circle: OA, OB, OC, etc., dividing the polygon into n $\triangle$ with O as vertex.	**1.** Why?
2. These $\triangle$ are all equal $\triangle$.	**2.** s. s. s. = s. s. s.
3. Every triangle has its alt. equal to r.	**3.** (178).
4. $\triangle AOB = \frac{1}{2} (AB) \times r$.	**4.** (358).
5. $S = n \times \triangle AOB$.	**5.** Why?
6. $S = \frac{1}{2} \times n \times (AB) \times r$.	**6.** Sub.
7. But $n \times (AB) = P$.	**7.** Why?
8. $\therefore S = \frac{1}{2} P \times r$.	**8.** Sub.

Ex. 1. Find the area of a square circumscribed about a circle of radius 12″.

Ex. 2. Find the area of a regular hexagon having a side of 6″.

PROPOSITION XIV. THEOREM

427. *The area of a circle is equal to one half the product of its circumference and radius.*



Given S the area of a circle of radius R and circumference C.

To prove $S = \frac{1}{2} C \times R$.

Proof

STATEMENTS	REASONS
1. Circumscribe a regular polygon about the circle, and let P be its perimeter, A its area.	1. Why?
2. $A = \frac{1}{2} P \times R$.	2. (426).
3. Let the number of sides be increased indefinitely, then	3. (420).
$A \longrightarrow S$.	
4. But $P \longrightarrow C$.	4. (416).
5. $\frac{1}{2} P \times R \longrightarrow \frac{1}{2} C \times R$.	5. (415 A).
6. $\therefore S = \frac{1}{2} C \times R$.	6. (414).

Alternate Proof

1. Same as (1) above.	1. Why?
2. $A = \frac{1}{2} P \times R$.	2. (426).
3. $S = \frac{1}{2} C \times R$.	3. (417).

428. COR. 1. *The area of a circle is equal to π times the square of its radius.*

For $S = \frac{1}{2} C \times R$ and $S = \frac{1}{2}(2\pi R)R = \pi R^2$.

429. Cor. 2. *The areas of circles are to each other as the squares of their radii, or of their diameters.*

430. Cor. 3. *The area of a sector whose central angle is $n°$ is equal to* $\frac{n}{360} \times \pi R^2$.

431. Cor. 4. *Similar sectors are to each other as the squares of their radii.*

Ex. 1. Construct a circle whose circumference shall equal the sum of the circumferences of two given circles.

Hint. Let a and b be the radii of the given circles and let x equal the radius of the required circle.

Then $2\pi x = 2\pi a + 2\pi b = 2\pi(a + b)$

or $x = a + b$.

Ex. 2. Construct a circle whose area shall equal the sum of the areas of two given circles.

Hint. $\pi x^2 = \pi a^2 + \pi b^2 = \pi(a^2 + b^2)$

or $x^2 = a^2 + b^2$. See § 368, (8).

Ex. 3. Construct a circle whose circumference equals the difference of the circumference of two given circles.

Ex. 4. Construct a circle that has its area equal to the difference between the areas of two given circles.

Ex. 5. Construct a circle having its area three times that of a given circle.

Ex. 6. Construct a circle so that its circumference equals twice that of a given circle.

Ex. 7. Construct a circle having its area equal to the area bounded by two given concentric circles.

Ex. 8. Construct a circle whose circumference shall be equal to the semicircumference of a given circle.

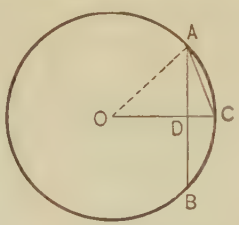
Ex. 9. Find the difference of the areas of the circles circumscribed about and inscribed in a square.

Ex. 10. What is the radius of a circle having an area of 616 sq. in.? ($\pi = \frac{22}{7}$.)

Ex. 11. Find the area of a 32° sector in a circle of radius 7 in.

PROPOSITION XV. PROBLEM

432. *The side and the radius of a regular polygon being given, to compute the side of a regular polygon of the same radius and double the number of sides.*



Given AB equal to a side s of a regular polygon inscribed in $\odot O$ with radius $OA = R$.

Required. To compute AC, the side of a regular polygon of double the number of sides, inscribed in the same circle.

Construction. Draw CO meeting AB in D. (Why?)

OC bisects AB at right angles. (79)

$$\overline{AC}^2 = \overline{OA}^2 + \overline{OC}^2 - 2\,OC \times OD, \quad (334)$$

or $$\overline{AC}^2 = 2R^2 - 2R \times OD. \quad (\text{Sub.})$$

But $$\overline{OD}^2 = R^2 - \left(\frac{s}{2}\right)^2. \quad (330)$$

$$\therefore OD = \sqrt{R^2 - \frac{s^2}{4}},$$

or $$OD = \tfrac{1}{2}\sqrt{4R^2 - s^2}.$$

Hence $$\overline{AC}^2 = 2R^2 - R\sqrt{4R^2 - s^2}.$$

$$\therefore AC = \sqrt{2R^2 - R\sqrt{4R^2 - s^2}}.$$

433. COR. If $R = 1$, and the side of a polygon on n sides equals s_n, then

$$s_{2n} = \sqrt{2 - \sqrt{4 - s_n^2}}.$$

Proposition XVI. Problem

434. *To compute the ratio of the circumference of a circle to the diameter.*

Let $R = 1$, then $s_6 = 1$,
and according to (433), we have

		Length of Side	Length of Perimeter
s_{12}	$= \sqrt{2 - \sqrt{4 - 1^2}}$	$= 0.51764$	6.21166
s_{24}	$= \sqrt{2 - \sqrt{4 - (0.51764)^2}}$	$= 0.26105$	6.26526
s_{48}	$= \sqrt{2 - \sqrt{4 - (0.26105)^2}}$	$= 0.13081$	6.27870
s_{96}	$= \sqrt{2 - \sqrt{4 - (0.13081)^2}}$	$= 0.06534$	6.28206
s_{192}	$= \sqrt{2 - \sqrt{4 - (0.06534)^2}}$	$= 0.03272$	6.28291
s_{384}	$= \sqrt{2 - \sqrt{4 - (0.03272)^2}}$	$= 0.01636$	6.28312
s_{768}	$= \sqrt{2 - \sqrt{4 - (0.01636)^2}}$	$= 0.00818$	6.28317

Taking the last perimeter as the approximate value of the circumference,

$$\pi = \frac{C}{2R} = \frac{6.28317}{2} = 3.14159.$$

435. Method XXI. **Most computations referring to the circle involve the use of the two formulæ $C = 2\pi R$ and $S = \pi R^2$. If R is not given, compute R first.**

Ex. 1. Find the circumference of a circle whose radius is 5.

Ex. 2. The diameter of a circle equals 10 in. Find the length of an arc of (*a*) 60°, (*b*) 1°, (*c*) 5°.

Ex. 3. Find the area of a circle whose radius is 10.

Ex. 4. Find the radius of a circle whose area is 100 sq. in.

Ex. 5. Find the radius of a circle whose circumference equals m inches.

Ex. 6. Find the circumference of a circle whose area equals 100π.

Ex. 7. Find the area of a circle whose circumference equals 4.

Ex. 8. Find the circumference of a circle whose area equals S.

Ex. 9. Find the area of a circle whose circumference equals C.

Ex. 10. Find the radius of a circle equivalent to a square whose side equals 6.

Ex. 11. The circumference of a circle equals 10. Find the circumference of a circle having twice the area of the given circle.

Ex. 12. Two concentric circles have their circumferences equal to 30 and 40 respectively. Find the area bounded by the two circles.

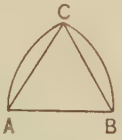
Ex. 13. Find a semicircle equivalent to an equilateral triangle whose side equals 5.

Ex. 14. Find the area of a sector whose radius equals 5, and whose central angle equals 40°.

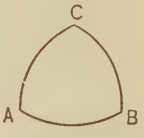
Ex. 15. A square is inscribed in a circle of radius 10. Find the area of a segment having a side of the square as its chord.

Ex. 16. The radius of a circle is 4. What is the area of a segment whose arc is (*a*) 120°, (*b*) 60°?

Ex. 17. In the diagram given here, $AB = BC = CA = 4$, B is the center of arc AC, and A is the center of arc BC. Find the total area.



Ex. 17



Ex. 18

Ex. 18. A is the center of arc BC, B the center of AC, and C the center of AB. Find the total area, if $AB = 6$ in.

Ex. 19. A circle has an area of 60 sq. ft. Find the length of an arc of 60°.

Ex. 20. Show how to inscribe a circle in a given sector.

Ex. 21. The radius of a circle is 20 ft.; the area of a sector of that circle is 25π sq. ft. What is its arc in degrees?

Ex. 22. The area of a sector is 18π; the angle of the sector is 80°. Find the radius of the sector.

Ex. 23. The altitude of an equilateral triangle is 6 in. Find the area of the circumscribed and inscribed circles.

Ex. 24. The areas of two circles are as 9 : 4; the radius of the larger is 6 in. Find the circumference of the smaller circle.

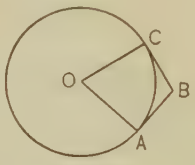
Ex. 25. Find the area of a circle inscribed in a square whose area is $12\frac{1}{4}$ sq. in.

Ex. 26. If the apothem of a regular hexagon is 4, what is the area of its circumscribed circle?

Ex. 27. It requires $41\frac{1}{7}$ rd. of fencing to inclose a semicircular field. Find the area of the field. (Assume $\pi = 3\frac{1}{7}$.)

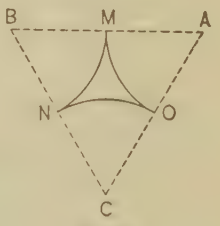
Ex. 28. Prove that the star-shaped polygon formed by producing the sides of a regular hexagon is equivalent to twice the hexagon.

Ex. 29. If AB and BC are tangents, $\angle COA = 60°$, and $OA = 10$ in. Find (*a*) area $OABC$, (*b*) area bounded by arc AC and the two tangents.



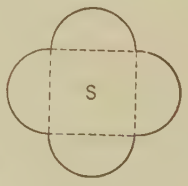
Ex. 29

Ex. 30. Find the area bounded by the arcs MN, NO, and OM, if each arc $= 60°$ and their common radius equals 10.



Ex. 30

Ex. 31. On the sides of a square whose side is 4 in., semicircles are constructed. Find the total area.



Ex. 31

Ex. 32. On the sides of a regular hexagon whose side equals 2 in., semicircles are drawn. Find the total area.

Ex. 32

Ex. 33. Find the radius of a circle whose circumference numerically equals its area.

Ex. 34. Find the central angle of a sector whose area is equal to the square of the radius.

Ex. 35. The area of a square inscribed in a circle is 16 sq. in. Find the area of the circle.

Ex. 36. Find the diameter of a wheel which, in going a mile, makes 500 revolutions.

MISCELLANEOUS EXERCISES

Ex. 1. Find the radius, the apothem, and the area of the following *regular* polygons:

(*a*) A square whose side equals 4
(*b*) A square whose side equals s
(*c*) A hexagon whose side equals 2
(*d*) A hexagon whose side equals s
(*e*) A triangle whose side equals 6
(*f*) A triangle whose side equals s

Ex. 2. A side of a regular circumscribed triangle is equal to twice a side of a regular inscribed triangle.

Ex. 3. The square of a side of an inscribed equilateral triangle is three times the square of a side of the regular inscribed hexagon.

Ex. 4. An equilateral polygon circumscribed about a circle is regular if the number of its sides is odd.

Ex. 5. An equiangular polygon inscribed in a circle is regular if the number of sides is odd.

Ex. 6. The area of an equilateral triangle is $36\sqrt{3}$ units. Find the area of the inscribed circle.

Ex. 7. Prove that an exterior angle of a regular polygon made by producing one side is equal to the angle at the center of the polygon.

Ex. 8. A radius of a circle is 12 in. Find the area and the perimeter of a segment of this circle whose chord equals the radius.

Ex. 9. Find the number of square inches of tin that would be wasted in cutting the largest possible circular disk from a piece in the form of an equilateral triangle 12 in. on a side.

Ex. 10. The side of a square is 12. Find the area included between the square and the circumscribed circle.

Ex. 11. A village has two standpipes each 50 feet high, one 10 feet in diameter and the other 16 feet in diameter. These two standpipes are to be replaced by one of the same height and capacity. Find the diameter of the new standpipe.

GENERAL REVIEW EXERCISES

Ex. 1. Prove that the medians drawn to the legs of an isosceles triangle are equal.

Ex. 2. The triangle ABC has a right angle at C and D trisects AC, so that $AD = 2\,DC$. It is found that $AD = BD$. Find the size of $\angle A$.

Ex. 3. Prove that two lines drawn from the extremities of a side of a triangle and limited by the other sides cannot bisect each other.

Ex. 4. Show how to divide a right triangle into two isosceles triangles.

Ex. 5. In a regular polygon of 15 sides, find the number of degrees in (1) each interior angle, in (2) each exterior angle.

Ex. 6. In a quadrilateral the diagonals are equal and also one pair of opposite sides are equal. Prove that two of the triangles into which the quadrilateral is divided by the diagonals are isosceles.

Ex. 7. The median of a trapezoid bisects the two diagonals.

Ex. 8. If a diagonal of a parallelogram bisects one angle, the figure is a rhombus.

Ex. 9. Draw a line terminated by the sides of a given angle, so that it shall be equal to a given line and parallel to another given line.

Ex. 10. Draw a circle concentric with a given circle and having a given chord in the given circle as a tangent.

Ex. 11. What is the locus of the vertex of a 60° angle the sides of which always pass through two fixed points?

Ex. 12. Three consecutive sides of an inscribed quadrilateral subtend arcs of 70°, 85°, and 98° respectively. Find each angle of the quadrilateral, an angle between its diagonals, and the angles formed by producing the opposite sides until they meet.

Ex. 13. Prove that if a trapezoid be inscribed in a circle, the nonparallel sides produced will meet on the perpendicular bisector of the parallel sides.

Ex. **14**. Find the locus of the centers of all circles of a given radius that cut off a given distance on a given line.

Ex. **15**. Upon a given line construct accurately with ruler and compasses a segment of a circle that contains an angle of 30°.

Ex. **16**. Given two diagonals of a regular pentagon intersecting within it. Find the number of degrees in an angle between them.

Ex. **17**. Through a point P within a circle a variable chord APB is drawn meeting the circle in A and B. Find the locus of the middle point of AB.

Ex. **18**. Show how to draw to a given circle two tangents which shall form a given angle.

Ex. **19**. If two polygons are similar to a third polygon, they are similar to each other.

Ex. **20**. From a point without a circle two tangents are drawn making with each other an angle of 60°. The length of each tangent is 15 inches. Find the diameter of the circle.

Ex. **21**. Two chords intersect within a circle; the segments of one are each 6, the total length of the other is 13. Find the length of its segments.

Ex. **22**. A rod 8 feet long is free to move within a rectangle 8 feet long and 6 feet wide. Describe accurately the boundary of the region within which the middle point of the rod will be found.

Ex. **23**. Construct an equilateral triangle given: (1) the radius of its inscribed circle; (2) the radius of its circumscribed circle.

Ex. **24**. The chord of a segment of a circle is 34 in. in length, and the radius of the circle is 60 in. Find the height of the segment.

Ex. **25**. The hypotenuse of a right triangle is 10 inches long and one angle is 30°. Find the lengths of the segments into which the shorter leg is divided by the bisector of the opposite angle.

Ex. **26**. The altitude of a triangle is 12 feet, and it divides the base into segments that are 5 feet and 9 feet. Find the perimeter of the triangle.

Ex. 27. Prove that if two equivalent triangles have two sides in each respectively equal then the included angles are either equal or supplementary.

Ex. 28. From a given point on a circle draw the chords that are bisected by a given chord. When is this impossible?

Ex. 29. The radii of two circles are 7 inches and 2 inches, and their centers are 13 inches apart. Find the length of their common tangents.

Ex. 30. Prove that if a point move about within a regular polygon, the sum of the perpendiculars let fall upon the sides (or the sides produced) will be constant.

Ex. 31. The bases of a trapezoid are 29 feet and 37 feet respectively, and its area is 247.5 square feet. Find its altitude.

Ex. 32. The sides of a triangle are 5, 5, and 6. Find (1) its area, (2) the area of its inscribed circle, (3) the altitude upon one of the sides 5, (4) the diameter of its circumscribing circle.

Ex. 33. The hypotenuse of an isosceles right triangle is 10 inches. What is the length of a side of the square whose area is three times that of the triangle?

Ex. 34. Given an irregular hexagon to construct an equivalent square.

Ex. 35. One diagonal of a rhombus is $\frac{8}{15}$ of the other, and the difference of the diagonals is 14. Find the area and the perimeter of the rhombus.

Ex. 36. In a circle of area 275 square feet there is inscribed a rectangle of 150 square feet. Find the lengths of the sides of the rectangle.

Ex. 37. The base of a triangle is 20 and its altitude is 30. Find the area of the trapezoid cut off by a line parallel to the base at a distance of 10 from it.

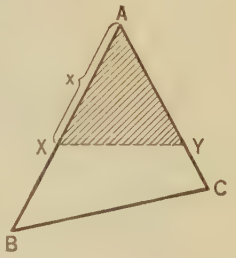
Ex. 38. What equal regular polygons will exactly cover a plane surface when they are of the same number of sides? when any combination is allowed?

Ex. 39. Express the area of a regular hexagon in terms of one of its sides (a).

SOLUTION OF PROBLEMS BY MEANS OF ALGEBRAIC ANALYSIS

436. If a problem requires the construction of lines, it is often possible to state the condition in the form of an equation. The solution of the equation gives the unknown line in an algebraic form, which may be constructed.

Ex. A. From a triangle ABC, to cut off an isosceles triangle AXY, equivalent to one half of ABC.



Analysis. Let

$$AX = AY = x,$$
$$AB = c,$$

and
$$AC = b.$$

$$\triangle AXY : \triangle ABC = x^2 : bc = 1 : 2. \quad \text{(Why?)}$$
$$\therefore 2x^2 = bc.$$
$$x = \sqrt{\frac{bc}{2}}.$$

Construction. Construct x, a mean proportional between $\frac{c}{2}$ and b, and on AB and AC respectively, lay off AX and AY equal to x.

Ex. B. From a point outside a circle, to draw a secant which is bisected by the circle.

Analysis. Let PAB be any secant. Suppose PCD is the required secant.

Then
$$PC = CD \text{ or } PD = 2\,PC.$$

Designate PB and PA by known letters b and a.

Then, by § 322,

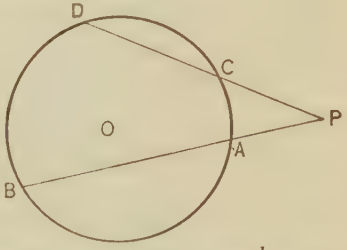
$$PC \cdot PD = PB \cdot PA$$

or, putting $PC = x$,

$$x \cdot 2x = b \cdot a,$$
$$2x^2 = ba,$$
$$x^2 = \frac{ba}{2} = \frac{b}{2} \cdot a.$$

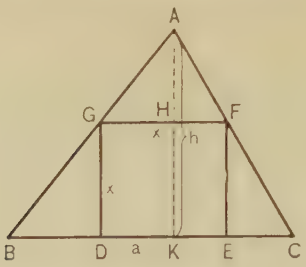
This becomes

$$\frac{b}{2} : x = x : a.$$



Construction. Construct x, a mean proportional between $\frac{b}{2}$ and a.

Ex. C. To inscribe a square in a given triangle.



Given $\triangle ABC$ to inscribe a square in this $\triangle$.

Analysis. Let x = a side of the required square.

Let the altitude $AK = h$

and the base $BC = a$.

By similar $\triangle$s, $BC : GF = AK : AH$,

or $a : x = h : h - x$.

Then $xh = ah - ax$,

and $$x = \frac{ah}{a + h} = \frac{a \cdot h}{(a + h)}.$$

In the form of a proportion, $a + h : a = h : x$.

Construction. Construct x as the fourth proportional to $a + h$, a, and h.

Ex. **1.** To divide the longer side of a rectangle into two parts so that the difference of their squares equals the area of the rectangle.

Ex. **2.** In a given square, to inscribe another square, having a given side.

Ex. **3.** To inscribe a square in a semicircle.

Ex. **4.** To transform a given square into a rectangle having a perimeter equal to twice the perimeter of the given square.

Ex. **5.** Given two concentric circles, to draw a chord in the larger circle so that it equals twice the chord formed in the smaller circle.

Ex. **6.** In the line AB, to find a point C so that $\overline{AC}^2 = 3\,\overline{CB}^2$.

Ex. **7.** To construct a square equivalent to the difference between a given square diminished by one half a second given square.

Ex. **8.** Through a given point within a circle, to draw a chord so that the given point is a trisection point of the required chord.

MAXIMA AND MINIMA OF PLANE FIGURES

437. DEF. The greatest of all magnitudes that satisfy given conditions is called the **maximum**; the least is called the **minimum**.

438. DEF. **Isoperimetric figures** are those which have **equal perimeters.**

PROPOSITION I. THEOREM

439. *Of all triangles having given two sides, that in which those two sides are perpendicular to each other is a maximum.*

Given in $\triangle ABC$ and DFE, $AB = DF$, $AC = DE$, $\angle A$ = rt. $\angle$, and $\angle D$ oblique.

To prove area $ABC >$ area DEF.

Proof. From E draw $EG \perp DF$.

Then $EG < DE$, (Why?)

or $EG < AC$.

$\therefore$ $\triangle ABC$ and DEF have equal bases and unequal altitudes.

$\therefore \triangle ABC > \triangle DEF$. (Why?)

$\therefore \triangle ABC$ is a maximum.

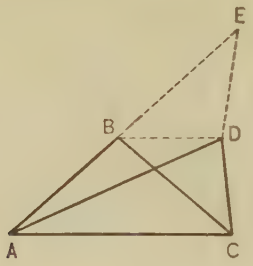
Ex. **1.** Of all triangles having given the base and the median to the base, which is the maximum?

Ex. **2.** Of all parallelograms having given the diagonals, which has the maximum area?

Ex. **3.** To divide a given line into two parts such that the rectangle contained by the segments is a maximum.

Proposition II. Theorem

440. *Of all triangles having the same base and equal areas, the isosceles triangle has the minimum perimeter.*



Given $\triangle ADC = \triangle ABC$, and $AB = BC$.

To prove $AB + BC + AC < AD + DC + AC$.

Proof. Produce AB by its own length to E. Draw BD and DE.

$BD \parallel AC$

(*for otherwise* $\triangle ABC$ *would not be* $= \triangle ADC$).

$\angle EBD = \angle BAC$. (Why?)

$\angle BAC = \angle BCA$. (Why?)

$\angle BCA = \angle CBD$. (Why?)

$\therefore \angle EBD = \angle CBD$. (Ax. 1.)

But $BC = BE$, (Why?)

and BD is common.

$\therefore \triangle BDC \cong \triangle BDE$.

$\therefore DC = DE$.

But $AD + DE > AB + BE$. (Why?)

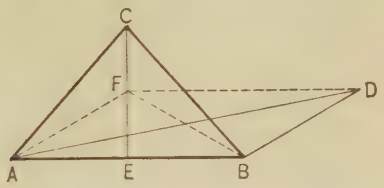
$\therefore AD + DC > AB + BC$,

or adding AC,

$$AC + AD + DC > AB + BC + AC.$$

PROPOSITION III. THEOREM

441. *The maximum of isoperimetric triangles on the same base is the one whose other two sides are equal.*



Given $\triangle ABC$ and ABD having equal perimeters, the common base AB, and $AC = CB$.

To prove area $ACB >$ area ADB.

Proof. Draw median CE, and $DF \parallel AB$, meeting CE in F. Draw FA and FB.

Then CE is the perpendicular bisector of AB. (Why?)

$$\therefore AF = FB.$$

But $\triangle AFB = \triangle ADB.$ (359)

$\therefore$ perimeter $AFB <$ perimeter $ADB.$ (440)

$$\therefore AF + FB < AD + DB.$$

But $AB + DB + DA = AB + BC + CA.$ (Hyp.)

$$\therefore DB + DA = BC + CA. \quad \text{(Ax. 3.)}$$

$$\therefore AF + FB < BC + CA. \quad \text{(Sub.)}$$

But $AF = FB$ and $BC = CA.$

Hence $AF < AC.$

$$\therefore FE < CE. \quad (330)$$

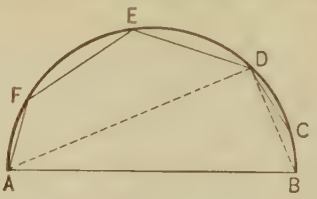
$\therefore$ area $AFB <$ area $ACB,$ (361)

or area $ADB <$ area $ACB.$

442. COR. *Of all isoperimetric triangles, the equilateral has the maximum area.*

Proposition IV. Theorem

443. *Of all polygons having all sides given in order but one, the polygon having the maximum area can be inscribed in a semicircle having the undetermined side as diameter.*



Given polygon $ABCDEF$, the maximum of all polygons, having given the sides AF, FE, ED, DC, and CB.

To prove $ABCDEF$ can be inscribed in a semicircle whose diameter is AB.

Proof. Join any vertex, as D, with A and B.

Then $\triangle ADB$ must be the maximum of all triangles that can be formed with sides AD and DB; *i.e.* $\angle ADB$ must be a rt. $\angle$. For otherwise by making $\angle ADB$ a right one without changing the sides AD and DB, we could increase $\triangle ADB$ without altering the remaining parts $AFED$ and DCB of the polygon. Or polygon $ABCDEF$ would be increased, which is contrary to the hypothesis, since $ABCDEF$ is a maximum.

$\therefore$ $\angle ADB$ is a right angle,

and D is on a semicircumference that can be constructed on AB.

For the same reason every vertex of the polygon must lie on the semicircumference.

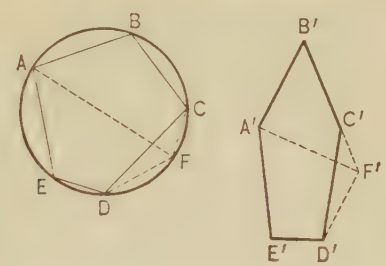
Note. We assume (1) a maximum, and (2) but one maximum.

Ex. 1. In quadrilateral $ABCD$, $AB = BC = CD =$ a given line. Determine the angles A, B, C, and D so that $ABCD$ becomes a maximum.

Ex. 2. To inscribe an angle in a given semicircle so that the sum of its arms is a maximum.

Proposition V. Theorem

444. *Of all polygons constructed with the same given sides in a given order, that which can be inscribed in a circle has the maximum area.*



Given polygon $ABCDE$ inscribed in a circle, mutually equilateral with polygon $A'B'C'D'E'$, which cannot be inscribed in a circle.

To prove area $ABCDE >$ area $A'B'C'D'E'$.

Proof. From A draw diameter AF, and join F to the two nearest vertices C and D.

On $C'D'$, construct $\triangle C'D'F'$ congruent to $\triangle CDF$, and join $A'F'$.

Area $ABCF >$ area $A'B'C'F'$. (443)

Area $FDEA >$ area $F'D'E'A'$. (443)

$\therefore$ area $ABCFDE >$ area $A'B'C'F'D'E'$.

But $\triangle CFD \cong \triangle C'F'D'$.

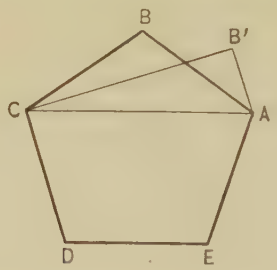
Then area $ABCDE >$ area $A'B'C'D'E'$. (Ax. 5.)

Ex. **1.** In quadrilateral $ABCD$, $AB = 2$ inches, $BC = CD = DA = 1$ inch. What must be the angles A, B, C, and D to make the area $ABCD$ a maximum?

Ex. **2.** Determine the angles of pentagon $ABCDE$, if its area is a maximum and $AB = BC = CD = 2$ inches, and $DE = EA = 2\sqrt{2}$ inches.

Proposition VI. Theorem

445. *Of all isoperimetric polygons of the same number of sides, the equilateral has the maximum area.*



Given $ABCDE$ the maximum of all isoperimetric polygons of the same number of sides.

To prove $AB = BC = CD = DE = EA$.

Proof. Suppose polygon $AB'CDE$, containing the two unequal sides CB' and AB', were the maximum.

Then construct on the diagonal AC the isosceles triangle $AB'C$, isoperimetric with ABC. (441)

Area $AB'C$ is less than area ABC.

$\therefore$ polygon $AB'CDE$ is less than the isoperimetric polygon $ABCDE$.

Hence polygon $AB'CDE$ cannot be the maximum.

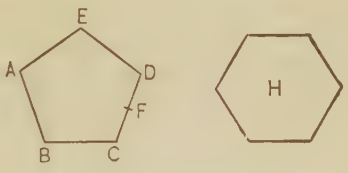
Or in the maximum polygon $AB = BC$. Therefore all sides of the maximum polygon are equal.

446. Cor. *Of all isoperimetric polygons of the same number of sides,* **the maximum is regular.** (444, 445)

Ex. In a given semicircle to inscribe a trapezoid whose area is a maximum.

Proposition VII. Theorem

447. *Of two isoperimetric regular polygons, that which has the greater number of sides has the greater area.*



Given regular pentagon *ABCDE* isoperimetric with regular hexagon *H*.

To prove area *H* > area *ABCDE*.

Proof. Let *F* be any point in *CD*.

ABCFDE may be considered a hexagon, having one of its angles equal to a straight angle.

ABCFDE is not an equilateral polygon. *H* is an equilateral polygon.

$\therefore$ area *H* > area *ABCDE*. (445)

448. Cor. *The area of a circle is greater than the area of any polygon whose perimeter equals the circumference of the circle.*

Ex. 1. In a given segment to inscribe an angle so that the sum of its arms is a maximum.

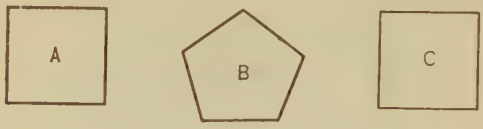
Ex. 2. In a given circle to inscribe a triangle, having a maximum perimeter.

Ex. 3. The circumscribed regular polygon has a smaller area than any other polygon circumscribed about the same circle.

Ex. 4. Of all equivalent parallelograms on the same base, which has the minimum perimeter?

Proposition VIII. Theorem

449. *Of two equivalent regular polygons, that which has the greater number of sides has the smaller perimeter.*



Given square A = regular pentagon B.

To prove perimeter of $A >$ perimeter of B.

Proof. Construct square C isoperimetric with B.

Area $C <$ area B, (447)

or area $C <$ area A.

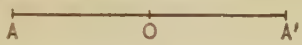
$\therefore$ perimeter of $C <$ perimeter of A.

$\therefore$ perimeter of $B <$ perimeter of A.

450. Cor. *The circumference of a circle is less than the perimeter of any equivalent polygon.*

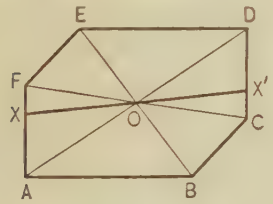
SYMMETRY

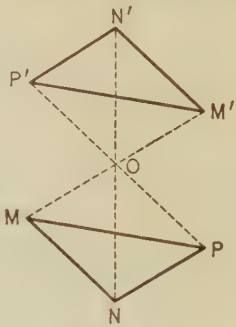
451. Two points A and A' are **symmetric with respect to a point** O, if O is the mid-point of AA'.



452. A figure is **symmetric with respect to a point** O, if any point in the figure is symmetric to another point in the figure with respect to O.

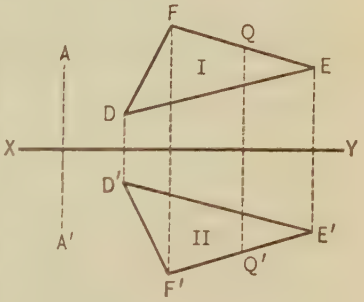
Thus, $ABCDEF$ is a figure symmetric to O. Obviously any line drawn through O and terminated by the figure, as XOX', is bisected. If the figure consists of two parts, as $\triangle MNP$ and $\triangle M'N'P'$, the two parts (*i.e.* the two triangles) are symmetric with regard to O.



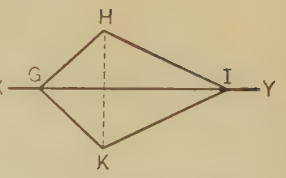


453. Two points A and A' are **symmetric with regard to an axis** XY, if XY is the perpendicular-bisector of AA'.

454. Two figures are **symmetric with regard to an axis** XY, if to every point in the figure Q, there exists a corresponding point in the other, Q', so that Q and Q' are symmetric with respect to XY.



Thus, $\triangle DEF$ and $\triangle D'E'F'$ are symmetric with respect to XY. The two figures may also form parts of one figure, as $GHIK$. Then the entire figure $GHIK$ is said to be symmetric with respect to XY.

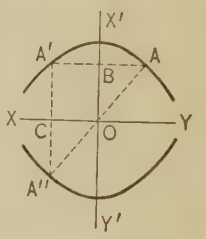


455. If two figures I and II are symmetric with regard to an axis XY and one figure I is turned about XY until it lies on the same side of XY as the other figure II, then I and II will coincide.

Conversely, if such a rotation produces a coincidence of the figures, they are symmetric with respect to XY.

456. THEOREM. *If a figure is symmetric with respect to two axes perpendicular to each other, it is symmetric with respect to their point of intersection.*

Let A be a point in the figure. There must exist another point A' so that $AA' \perp X'Y'$ and $AB = A'B$. Similarly there must be another point A'' so that $A'A'' \perp XY$ and $A'C = A''C$. Obviously $AA' \parallel XY$ and $A'A'' \parallel XY$.



Draw AA''.

Then $X'Y'$ bisects AA''. (148)

Similarly XY bisects AA''. (148)

$\therefore$ XY and $X'Y'$ meet AA'' in its mid-point.

Or line AA'' is bisected by O, the point of intersection of XY and $X'Y'$. Or to any point (A) there exists another A'' so that A and A'' are symmetric with regard to O.

Ex. 1. A parallelogram is symmetric with respect to the point of intersection of its diagonals.

Ex. 2. An isosceles triangle is symmetric with respect to the median drawn to the base.

Ex. 3. Quadrilateral $ABCD$ is symmetric with respect to the diagonal AC, if $AB = AD$, and $BC = CD$.

Ex. 4. A regular hexagon is symmetric with respect to its center.

Ex. 5. Two figures symmetric with respect to an axis are congruent.

Ex. 6. Two figures symmetric with respect to a point are congruent.

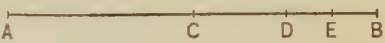
THE INCOMMENSURABLE CASE BASED UPON LIMITS[1]

457. A **constant** is a quantity that maintains the same value throughout the same discussion. A **variable** is a quantity whose value changes during the same discussion.

Two quantities are **commensurable** if they have a common measure. Two quantities are **incommensurable** if they have no common measure.

458. If a variable x approaches a constant a so that the difference between a and x becomes and remains less than any conceivable number (not zero), then a is called the **limit** of x.

For example, suppose a point P to move from A to B in such a way as to move in the first second over half of AB to C, in the second second, over half of the remainder, CB, to D, in the third second over half of the remainder, DB, to E, and so on indefinitely.



It is evident that the distance from A to the moving point P is a variable whose value can be made to differ from AB by less than any assigned quantity, although it never can be made equal to AB.

AB is therefore the **limit of the variable.**

Ex. (*a*) What is the limit of .999 $\cdots$?

(*b*) What is the limit of $6 + 3 + \frac{3}{2} + \frac{3}{4} + \cdots$ to infinity?

459. THEOREM. **If two variables, x and y, are always equal and x approaches a as a limit, then y approaches a as a limit.**

Proof. $a - x = a - y$, and $a - x$ can be made less than any assignable number.

Hence, $a - y$ can be made smaller than any assignable number.

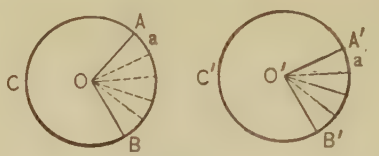
$$\therefore y \longrightarrow a.$$

460. COR. **If two variables are always equal and each approaches a limit, the limits are equal.**

[1] This section is given for teachers who wish to introduce limits in connection with Book II. Otherwise it should be omitted. See note, p. 124.

Proposition I. Theorem

461. *In the same circle, or in equal circles, two central angles have the same ratio as their arcs.*



Given in the equal circles ABC and $A'B'C'$, two central angles AOB and $A'O'B'$, having the arcs AB and $A'B'$ respectively.

To prove $$\frac{\angle AOB}{\angle A'O'B'} = \frac{\overset{\frown}{AB}}{\overset{\frown}{A'B'}}.$$

Proof. Case I. When the arcs are commensurable.

Let arc a be a common measure contained in $\overset{\frown}{AB}$ m times and in $\overset{\frown}{A'B'}$ n times.

Then, $$\frac{\overset{\frown}{AB}}{\overset{\frown}{A'B'}} = \frac{m}{n}.$$

Connect the points of division with the center. Then $\angle AOB$ will have been divided into m parts and $\angle A'O'B'$ into n, all being equal. (183)

Whence $$\frac{\angle AOB}{\angle A'O'B'} = \frac{m}{n}.$$

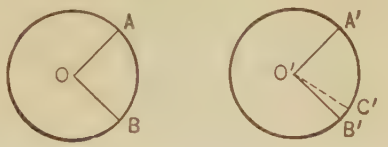
Hence if $$\frac{\overset{\frown}{AB}}{\overset{\frown}{A'B'}} = \frac{m}{n},$$

then $$\frac{\angle AOB}{\angle A'O'B'} = \frac{\overset{\frown}{AB}}{\overset{\frown}{A'B'}}.$$ (Ax. 1.)

Note. If the student cannot see the proof clearly by using the numbers m and n, let him use some such ratios as 4 to 5, and then generalize his results.

Case II. When the arcs are incommensurable.

Divide $\overset{\frown}{AB}$ into any number of equal parts, and apply one of those parts to $\overset{\frown}{A'B'}$ as many times as possible.



Since $\overset{\frown}{AB}$ and $\overset{\frown}{A'B'}$ are incommensurable, there must be a remainder $C'B'$ less than one of the equal parts.

Draw $O'C'$.

Since the arcs AB and $A'C'$ are commensurable,

$$\frac{\overset{\frown}{A'C'}}{\overset{\frown}{AB}} = \frac{\angle A'O'C'}{\angle AOB}.$$

By increasing the number of parts into which AB is divided, we can diminish the length of each part, and, therefore, the length of $C'B'$ indefinitely.

Hence $\overset{\frown}{A'C'}$ approaches $\overset{\frown}{A'B'}$ as a limit, and $\angle A'O'C'$ approaches $\angle A'O'B'$ as a limit.

Therefore

$$\frac{\overset{\frown}{A'C'}}{\overset{\frown}{AB}} \longrightarrow \frac{\overset{\frown}{A'B'}}{\overset{\frown}{AB}}, \text{ and } \frac{\angle A'O'C'}{\angle AOB} \longrightarrow \frac{\angle A'O'B'}{\angle AOB}.$$

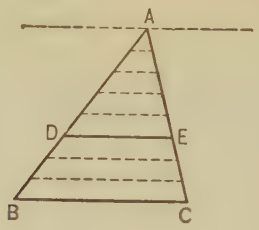
The variables $\dfrac{\overset{\frown}{A'C'}}{\overset{\frown}{AB}}$ and $\dfrac{\angle A'O'C'}{\angle AOB}$ being always equal, must have equal limits. (460)

Whence $$\frac{\overset{\frown}{A'B'}}{\overset{\frown}{AB}} = \frac{\angle A'O'B'}{\angle AOB}.$$

462. Cor. *A central angle is measured by its intercepted arc.*

Proposition II. Theorem

463. *A line parallel to one side of a triangle divides the other two sides into segments that are proportional.*



Given in $\triangle ABC$, DE parallel to BC.

To prove $AD : DB = AE : EC$.

Proof. Case I. AD and DB are commensurable.

Let m be a common measure contained in AD a times and in DB b times.

Then $$\frac{AD}{DB} = \frac{a}{b}.$$

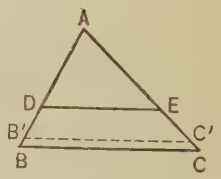
Through the points of division of AB draw parallels to BC. These lines divide AE into a parts and EC into b parts, all being equal. (148)

Whence $$\frac{AE}{EC} = \frac{a}{b}.$$

$$\therefore \frac{AD}{DB} = \frac{AE}{EC}. \qquad \text{(Ax. 1.)}$$

Case II. AD and DB are incommensurable.

Divide AD into any number of equal parts, and apply one of those parts to DB as many times as possible.



As the lines AD and DB are incommensurable, there must be a remainder, $B'B$, less than one of the equal parts.

Draw $B'C'$ parallel to BC.

The lines AD and DB' are commensurable.

$$\therefore \frac{AD}{DB'} = \frac{AE}{EC'}.$$

By increasing the number of parts into which AD is divided, we can diminish the length of these parts, and therefore the length of $B'B$ indefinitely.

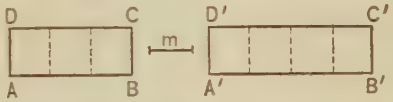
Hence DB' approaches DB as a limit, and EC' approaches EC as a limit.

The variables $\frac{AD}{DB'}$ and $\frac{AE}{EC'}$, being always equal, must have equal limits.

Whence
$$\frac{AD}{DB} = \frac{AE}{EC}. \tag{460}$$

Proposition III. Theorem

464. *Rectangles having equal altitudes are to each other as their bases.*



Given in rectangles $ABCD$ and $A'B'C'D'$, altitude AD = altitude $A'D'$.

To prove
$$\frac{ABCD}{A'B'C'D'} = \frac{AB}{A'B'}.$$

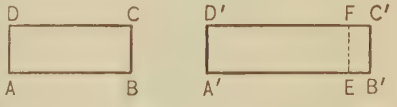
Proof. Case I. AB and $A'B'$ are commensurable.

Let m be a common measure contained in AB a times and in $A'B'$ b times.

[To be completed by the student. Compare (463).]

Case II. AB and $A'B'$ are incommensurable.

Divide AB into any number of equal parts, and lay off one of these parts on $A'B'$ as many times as possible.



As AB and $A'B'$ are incommensurable, there must be a remainder, EB', less than one of the equal parts.

Draw $EF \perp A'B'$.

As AB and $A'E$ are commensurable,
$$\frac{ABCD}{A'EFD'} = \frac{AB}{A'E}.$$

By increasing the number of parts into which AB is divided, we can diminish the length of these parts, and, therefore, the length of EB' indefinitely.

Hence, $A'E$ approaches $A'B'$ as a limit, and $A'EFD'$ approaches $A'B'C'D'$ as a limit.

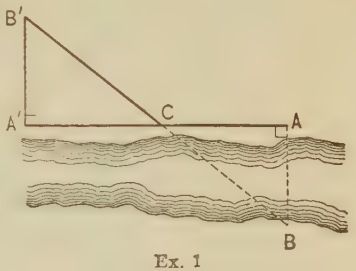
The variables $\frac{ABCD}{A'EFD'}$ and $\frac{AB}{A'E}$, being always equal, must have equal limits.

Whence $$\frac{ABCD}{A'B'C'D'} = \frac{AB}{A'B'}.$$

PRACTICAL APPLICATIONS OF PLANE GEOMETRY

BOOK I [1]

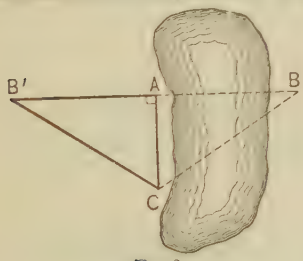
1. In order to determine the distance across a river AB, we measure AA' at right angles to AB, and place a stake at C, the mid-point of AA'. From A' we walk in a direction perpendicular to AA' until we reach the point B', which is in a straight line with B and C. Which distance must we measure in order to obtain the width of the river? Why?



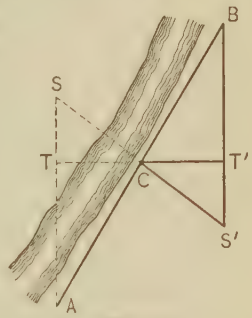
Ex. 1

2. Show what line we have to measure in order to determine the width of a lake AB, if we make $\angle ACB = \angle ACB'$ and $BB' \perp AC$.

3. In order to measure the distance of a tree from a steeple S, both on the other side of a river, I locate a point A



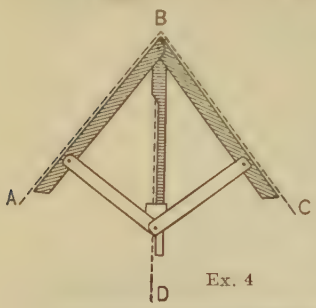
Ex. 2



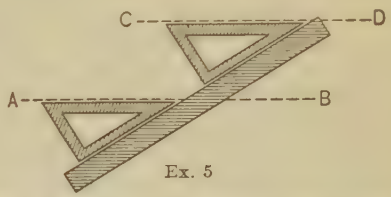
Ex. 3

[1] It is strongly recommended that students measure some angles in the field by means of a transit, a plane table, a sextant, etc. It is not necessary that such instruments be very elaborate, and even very crude ones, made of two paper protractors, a ruler, and a few pins, are serviceable.

in the prolongation of ST, take a line AB, and place a stake at its mid-point C. From B, I walk in a direction BS' so that $\angle B = \angle A$, and put stakes at point T' which lies in the prolongation of TC, and at the point S', which lies in the prolongation of SC. Which line must I measure? Prove your statement.



Ex. 4

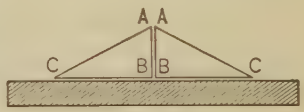


Ex. 5

4. Show how the annexed *Angle Divider* can be used to obtain the bisector BD of any angle ABC.

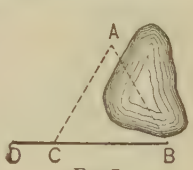
5. Show how a ruler and a triangle can be used to draw a line CD parallel to a given line AB.

6. A triangle is placed against a ruler in the two positions which are indicated in the diagram, and the position of AB is in both cases the same. How many degrees are in angle B?

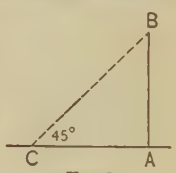


Ex. 6

7. To measure the distance AB, we walk from B toward D so that $\angle B = 60°$, until we reach C, a point at which $\angle ACB = 60°$. What line must we measure to obtain AB? Why?



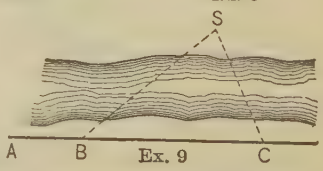
Ex. 7



Ex. 8

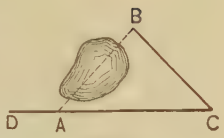
8. At a point C, 80 ft. from the foot of a pole AB, the angle ACB was found to be 45°. How high is the pole?

9. Walking along the bank of a river from A to C, a surveyor measures at B the angle ABS, and walking 800 ft. further to C, he found angle BCS to be one half of angle ABS. What is the distance BS?



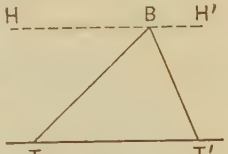
Ex. 9

10. Find the length AB of a lake from the following measurements: $\angle DAB = 138°$, $\angle C = 42°$, $AC = 610$ yd., $BC = 400$ yd.



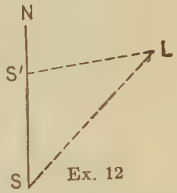
Ex. 10

11. A man in a balloon B measures the angles which a horizontal line HH' forms with the lines drawn to two towns T and T' as follows: $\angle HBT = 40°$, $\angle H'BT' = 70°$. If the distance between the towns is two miles, and if the balloon is directly above the line TT', what is the distance of the balloon from T?



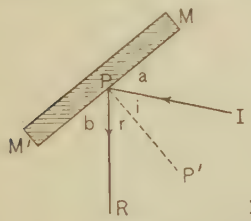
Ex. 11

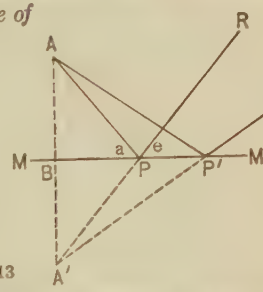
12. A ship sailing north at the rate of 10 mi. per hour occupied at 8 A.M. the position S, and at 10 A.M. the position S'. How far was the ship at 10 A.M. from a lighthouse L, if $\angle S = 43°$ and $\angle NS'L = 86°$?



Ex. 12

13. In the diagram given here (and in the diagram of similar exercises) MM' represents a mirror perpendicular to the plane of the paper, and all other lines represent lines *in* the plane of paper. If $P'P$ is $\perp M'M$, IP is a ray of light and PR is the reflected ray, then $\angle i$ (*angle of incidence*) always equals $\angle r$ (*angle of reflection*), hence $\angle a = \angle b$.



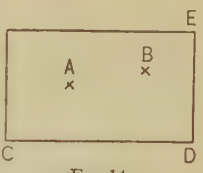


Ex. 13

If rays emitted from a point A are reflected by a mirror MM', prove that we obtain the reflected rays (*i.e.* make the necessary angles equal) by the following construction: Draw $AB \perp MM'$ and produce AB by its own length to A'. Draw $A'P$. PR, the prolongation of AP, is the reflected ray.

14. A billiard ball is reflected by a cushion according to the same law as a ray of light is reflected by a mirror, *i.e.* the angle of incidence equals the angle of reflection. (Ex. 13.)

The billiard ball A is to hit a point P in the cushion CD, and be reflected therefrom so as to strike B. Determine the location of P by a construction.



Ex. 14

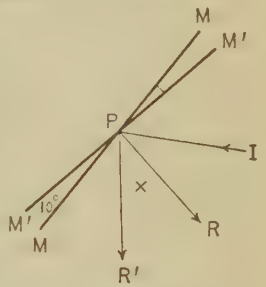
15. If the ball A shall first strike cushion CD, then cushion DE, and finally hit B, construct the path of the billiard ball.

16. The ray IP is reflected by MM in the direction PR, and if the mirror is turned 10° in the position $M'M'$ ($\angle MPM' = 10°$), the same ray is reflected in the direction PR'. Find $\angle RPR'$.

17. Solve the same problem if

$$\angle MPM' = n°.$$

NOTE. The result of this problem forms the principle of the sextant, an instrument used chiefly by sailors, for measuring angles.



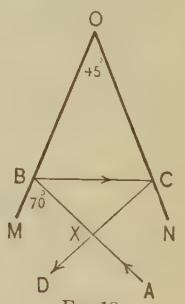
Ex. 16

18. Two mirrors MO and NO form an angle O equal to 45°. If a ray AB is reflected in the direction BC, and BC in the direction of CD, find the angle formed by the first and last ray (*i.e.* $\angle x$), if

(*a*) $\angle ABM = 70°$.

(*b*) $\angle ABM = n°$.

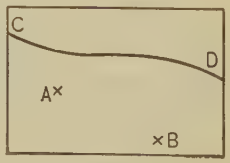
NOTE. The preceding exercise explains the principle of an instrument (called *optical square*) which is used to lay off right angles.



Ex. 18

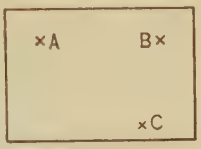
19. The annexed diagram represents a portion of a map, A and B two towns, and CD a railroad.

It is proposed to build a station, S, which is just as far from A as from B. Locate S by a construction.



Ex. 19

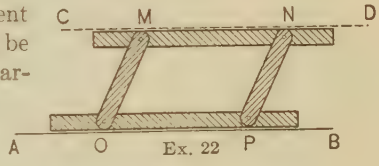
20. A, B, and C represent the locations of three towns (on a map). It is proposed to build a common schoolhouse, S, which is equally far from A, B, and C. By a construction locate S.



Ex. 20

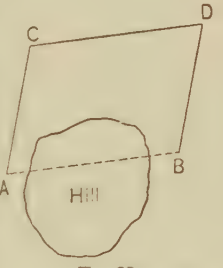
21. From a ship which sails N.E. (*i.e.* northeast) at the rate of 6 mi. an hour, a lighthouse is observed. At 9 P.M. the lighthouse appears exactly East, at 11 P.M. it appears exactly South. At what hour was the ship nearest to the lighthouse and what was the distance between them at this hour?

22. Explain how the instrument represented in the figure can be used to draw lines (*e.g.* CD) parallel to a given line AB [M, N, O, and P are pivots, $MO = NP$, and $MN = OP$].

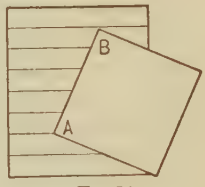


Ex. 22

23. To measure the distance from A to B, two points on opposite sides of a hill, lay off two lines, AC and BD, both running due north, and make $AC = BD$. Which line must be measured to obtain the length of AB? Why?



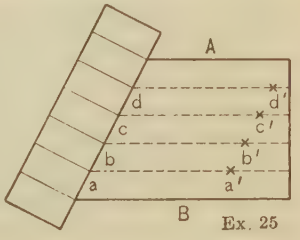
Ex. 23



Ex. 24

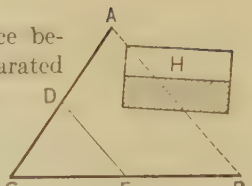
24. Explain how an ordinary ruled sheet can be used to divide a given line AB in 5 equal parts. Under what conditions is it impossible to use this method?

25. To divide a strip AB into 5 equal strips by lines parallel to A and B, apply a ruled sheet as indicated in the figure and obtain the 4 points a, b, c, d. In like manner obtain a', b', c', and d', and join the corresponding points. Prove this construction.



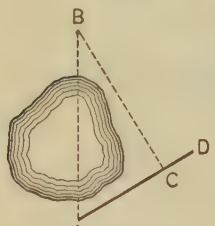
Ex. 25

26. Show how to determine the distance between two points A and B, which are separated by a house, by measuring the following lines: $AD = 9$ yd., $AC = 18$ yd., $EC = 11$ yd., $CB = 22$ yd., and $DE = 10$ yd.



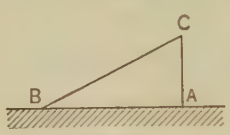
Ex. 26

27. To measure the distance AB, a man walks in the direction AD, so that $\angle BAD = 60°$, and stops at a point C, where $\angle BCD = 90°$. If $AC = 200$ yd., what is the length of AB?

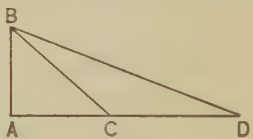


Ex. 27

28. The top of a flagstaff, 75 ft. high, is partly broken and touches the ground at B. Find the height of AC, if $\angle B = 30°$.



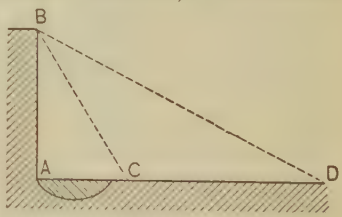
Ex. 28



Ex. 29

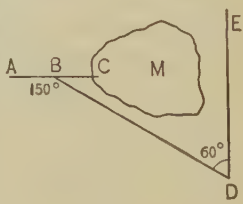
29. To measure the height, AB, of a tower, a man walks on a level straight road DCA that leads directly to the foot of the tower. At D, $\angle BDA = 15°$, and at C, $\angle BCA = 30°$. Find AB, if $DC = 300$ ft.

30. AB is a perpendicular cliff rising at one side of a river. An observer at a point C, 20 ft. from the opposite bank, measures $\angle ACB = 60°$, and walking 200 ft. further away from the river to D, he measures $\angle BDA = 30°$. Find the breadth of the river.



Ex. 30

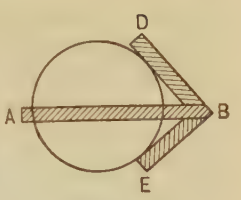
31. A straight railway AB meets a mountain at C, and a tunnel is being driven at C in the direction AC. It is proposed to commence this work at the other side of the mountain also. Angle ABD is made equal to 150°, $BD = 3$ km., and $\angle D = 60°$. How far from D in DE must the tunnel be driven, and what is its direction?



Ex. 31

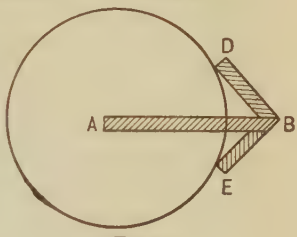
BOOK II

32. The figure represents an instrument used for locating the center of circular disks. Three pieces of metal are so joined that the edge AB bisects the angle formed by BD and BE, BD and BE being equal. Prove that AB passes through the center of the circle.



Ex. 32

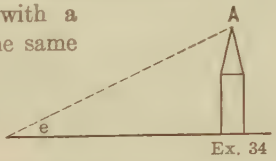
33. The next figure represents the same instrument applied to a much larger disk. Prove that AB passes through the center.

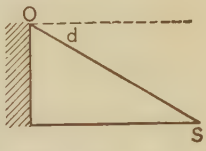


Ex. 33

34. The *angle of elevation* or *angle of depression* of any object A as observed from O, is the angle which the line from the eye of the observer O to the object makes with a horizontal line in the same vertical plane.

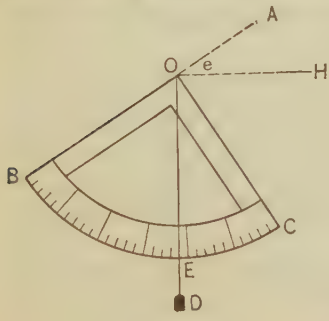
If the object lies higher than the observer, the angle is an angle of elevation, as $\angle e$. If the object lies lower than the observer, the angle is an angle of depression, as $\angle d$.

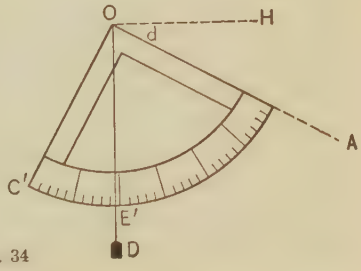




Ex. 34

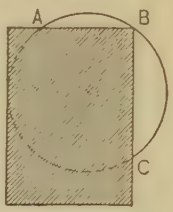
Prove that the angle of elevation of an object may be measured as follows: A quadrant is placed in vertical position so that the pro-





Ex. 34

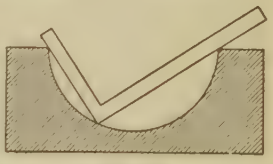
longation of BO passes through A. (This is done by means of "sights" attached at BO.) A plumb line OD, held by a pin at O, intersects the arc BC at E. The angle of elevation is measured by arc EC. Similarly arc $C'E'$ measures the angle of depression d.



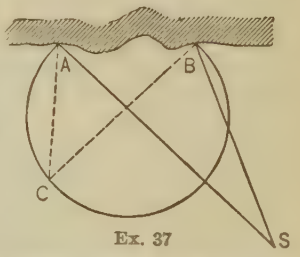
Ex. 35

35. Prove that a diameter of a circular disk may be drawn from A as follows: Place a rectangular sheet of paper so that one vertex, B, lies on the circle, and the adjacent side passes through A. C is the other end of the required diameter.

36. Justify the following method for testing the accuracy of a semicircular groove: Place a carpenter's square in the position indicated in the figure. If its vertex touches every point of the groove as the square slides around, the groove is a true semicircle.


Ex. 36

37. If A and B are two lighthouses, and dangerous rocks lie within the circle ABC, while all points without the circle are navigable, prove that a ship S is not in danger of striking the rocks as long as $\angle S$ is less than $\angle C$.



Ex. 37

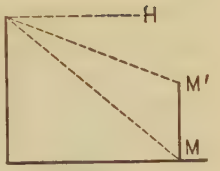
NOTE. The following six constructions (38 to 43) should be carried out by means of protractor and ruler. The required lines, or angles, should be found (approximately) by measurement. (See p. 303, Ex. 19.)

38. A vertical pole 25 ft. high casts a shadow 40 ft. long. What is the angle of elevation of the sun? (Let 1 in. represent 20 ft.)

39. A perpendicular cliff rises from the bank of a river. The angle of depression of the other bank, as observed from the top of the cliff, is 40°. If the cliff is 120 ft. high, what is the breadth of the river? (Let 1 in. represent 60 ft.)

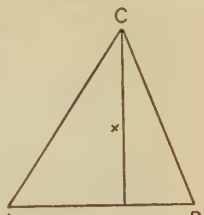
40. At a horizontal distance of 80 ft. from the foot of a tower, the angle of elevation of its top is 50°. Find the height of the tower.

41. From a window 60 ft. above the ground, the angles of depression of the top and the bottom of a monument are observed to be 20° and 40°. Find the height of the monument.



Ex. 41

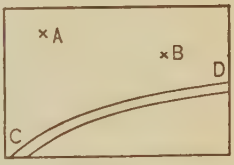
42. From two stations A and B, on opposite sides of a cloud, the angles of elevation are found to be $A = 60°$, $B = 70°$. Find the height of the cloud if $AB = 2$ mi.



Ex. 42

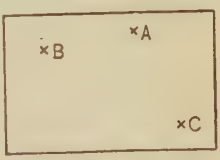
43. If the radius of the equator is 4000 mi., what is the radius of the parallel 40° N.? (Make 2000 mi. = 1 in.)

44. On a map CD represents a road and A and B two steeples. A man walking on the road from C to D measured at a certain point X, the angle $AXB = 45°$. He also noticed that X was nearer to B than to A. Locate the point X on the map.



Ex. 44

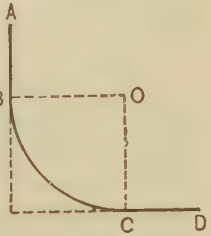
45. A, B, and C are three points on a map. At a certain point X, a surveyor measured $\angle AXB = 90°$ and $\angle AXC = 60°$. Locate the point X on the map.



Ex. 45

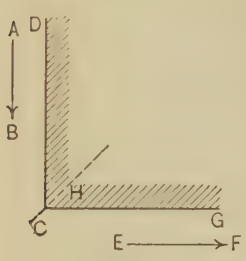
46. At the rectangular intersection of two streets the corner of the sidewalk is sometimes "rounded off." Construct the arc representing the rounded corner if its radius equals 25 ft. and if we assume the scale 1 : 500 (*i.e.* in the drawing 1 in. represents 500 in.).

47. Solve the preceding problem if the streets intersect at an acute angle.

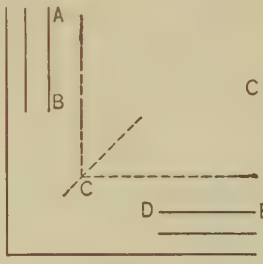


Exs. 46, 47

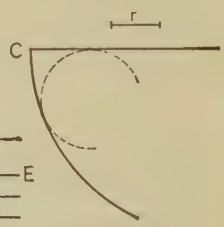
48. A wheelman riding in the direction AB ($\| DH$) wishes to turn the corner by moving in a circular path so that the nearest approach to the sidewalk is indicated by the point C. He continues his ride in the direction EF ($\| HG$). If CH bisects $\angle H$, and the distance of the first parallel lines (AB and DH) is equal to the distance of the second pair (HG and EF), construct the arc representing his path.



Ex. 48



Ex. 49

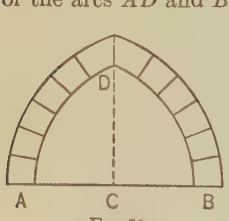


Ex. 50

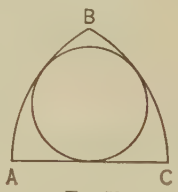
49. Solve the analogous problem for the track of a street car line.

50. "Round off" the corner C, made by a straight street and a circular one, by constructing a circle whose radius equals a given line r.

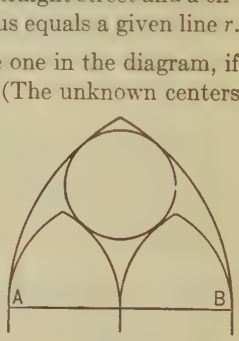
51. Construct a Gothic arch similar to the one in the diagram, if the span AB and the height CD are given. (The unknown centers of the arcs AD and BD lie in AB.)



Ex. 51



Ex. 52



Ex. 53

52. A Gothic arch ABC is equilateral, if A is the center of BC and C is the center of AB.

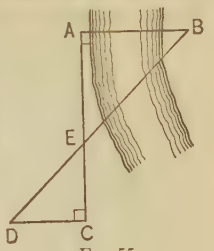
Inscribe a circle in an equilateral arch.

53. Construct the annexed figure, which consists of three equilateral arches and a circle touching the three, if AB is given.

BOOK III

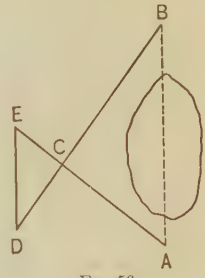
54. The shadow of a steeple upon level ground is 120 ft., while a pole 8 ft. long casts a shadow 3 ft. long. How high is the steeple?

55. To determine AB, the width of a river, measure AC at right angles to AB, and CD at right angles to AC. By sighting from D to B locate the point E. What is the length of AB if $AE = 200$ ft., $EC = 25$ ft., $DC = 20$ ft.?



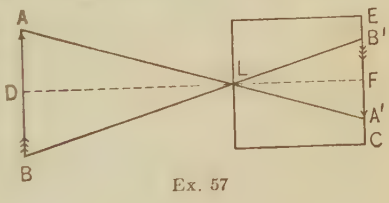
Ex. 55

56. Show how the distance AB, which cannot be measured directly, may be determined by the following measurements: $AE = 80$ ft., $EC = 20$ ft., $DB = 100$ ft., $DC = 25$ ft., $ED = 30$ ft.



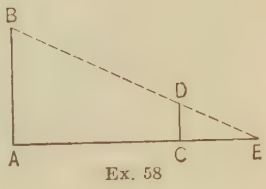
Ex. 56

57. The location of the image A' of a point A, formed in a photographer's camera, is approximately found by drawing a straight line AA' through the center of the lens L. If CE is the position of the photographic plate, then $A'B'$ is the image of AB. How large is $A'B'$ if $AB = 6$ ft., $LD = 12$ ft., and $LF = 6$ in.?



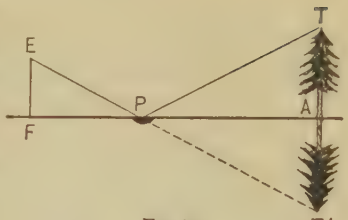
Ex. 57

58. The top of a tower AB appears in a straight line with D, the top of a pole DC, when viewed from E. If EA is horizontal, $EC = 4$ ft., $DC = 3$ ft., and $CA = 36$ ft., what is the height of the tower?



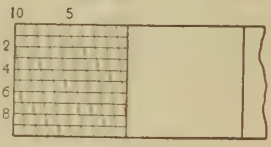
Ex. 58

59. An observer sees the reflected image of the top of a tree formed by a pool of water in the direction ET'. If FPA is a horizontal straight line, both EF and TA are perpendicular to FPA, $EF = 5$ ft., $FP = 8$ ft., and $PA = 30$ ft., find the height of the tree.



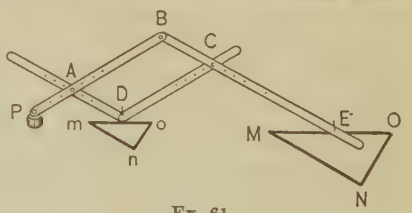
Ex. 59

60. If the side of the square in the diagram (*diagonal square*) is unity, show how we may measure .1, .2, .65, .34, etc.



Ex. 60

61. A *pantograph* is a machine for drawing a figure MNO similar to a given figure mno. It is used for enlarging and reducing maps and drawings. It consists of four bars, which are jointed at A, B, C, and D, and which are made parallel by making $AB = DC$ and $AD = BC$. P turns about a fixed pivot, and pencils are carried at D and E. The length of PA and CE is so adjusted that $\frac{PA}{AD} = \frac{DC}{CE}$. Prove that (1) P, D, and E are in a straight line, (2) $\frac{PD}{PE} = \frac{PA}{PB}$, (3) any straight line MN is parallel to mn and $\frac{mn}{MN} = \frac{PA}{PB}$, (4) any triangle MNO drawn by the pantograph is similar to the given triangle mno.



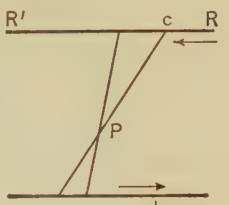
Ex. 61

62. The figure represents an instrument (*proportional compasses*) for obtaining a line CD which is any given fraction of a given line AB. By adjusting the length of OC, OD we may make this fraction equal to $\frac{2}{5}$ of AB, etc. Prove that $CD = \frac{2}{5}$ of AB, if $OC = \frac{2}{5}\ OB$ and $OD = \frac{2}{5}\ OA$.

Ex. 62

63. A map is drawn to the scale of 1 in. to 100 mi. How far apart are two places that are $3\frac{1}{2}$ in. apart on the map?

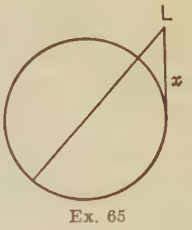
64. A man walking at the rate of 3 mi. per hour on a straight road ab which is parallel to a straight railroad RR' observes that a pole P appears in the same line of sight with a point C of a moving train. If the distance of P from ab is 10 ft. and the distance between RR' and ab is 100 ft., what is the rate of the moving train? Does it move at uniform speed?



Ex. 64

NOTE. The following examples which relate to the earth can often be simplified by bearing in mind that some numbers are only approximations. Thus, if a secant equals $8000\frac{99}{5250}$ mi., we may drop the fraction and make it equal to 8000, since the error in assuming the diameter of the earth as 8000 is much greater than the small fractions. Similarly we may measure the distance between two points on the earth's surface by a straight line instead of an arc, etc.

65. If the diameter of the earth is 8000 mi., how far can you see (*a*) from a lighthouse 96 ft. high? (*b*) From a mountain 2400 ft. high?



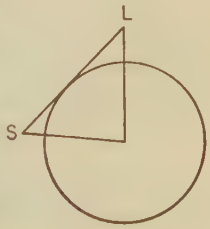
Ex. 65

66. Prove that n, the number of miles one can see at sea from an elevation of h ft., can be found approximately by the formula

$$n = \sqrt{\frac{3h}{2}}.$$

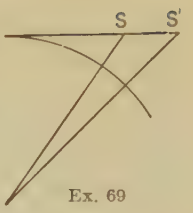
67. A swimmer places his eye at the surface of a smooth lake and finds that the top of a sailboat 4 mi. away is just visible. How high is the top of the sailboat above the water?

68. From the deck of a steamer 24 ft. above the sea, the top of a lighthouse that is 40 ft. high can just be seen beyond the horizon. Find the distance (SL) between steamer and lighthouse.



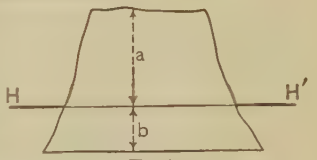
Ex. 68

69. From the deck of a steamer which is 54 ft. above sea level, the top of the mast of a sailboat appears in line with the horizon. If the sailboat is 2 mi. distant from the steamer, find the height of its mast.



Ex. 69

70. From the deck of a steamer which is 54 ft. above sea level, an iceberg is sighted and the height of its visible part is divided by the line of the horizon HH' into two parts a and b so that $a = 3b$. If the distance of the iceberg from the steamer is 3 mi., find the height of the iceberg.



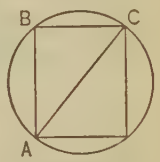
Ex. 70

71. From a point 6 ft. above sea level the visible horizon has a radius of 3 mi. Find the diameter of the earth.

72. The Harder funicular railway near Interlaken (Switzerland) has a length of 1593 yd., and an inclination of 30°. If the lower station has an elevation of 1900 ft. above sea level, find the elevation of the upper station.

Ex. 73

73. A skyrocket is seen to explode at an angle of elevation $A = 45°$. If the flash is seen $\frac{3}{4}$ sec. before the report of the explosion is heard, find the height of the skyrocket when it exploded. (Assume velocity of sound equal to 330 meters per second.)

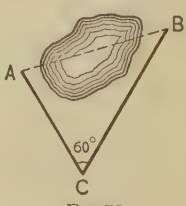


Ex. 74

74. The strongest beam that can be cut from a given round log is the one in which the height AB is to the width BC as $\sqrt{2}:1$. Draw a diameter AC and *construct* B so that

$$AB : BC = \sqrt{2} : 1.$$

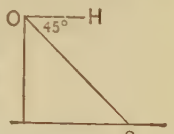
75. What is the distance between two points A and B on opposite sides of a hill if $AC = 20$ rd., $BC = 32$ rd., and $\angle C = 60°$?



Ex. 75

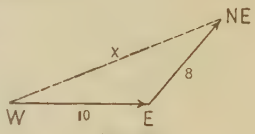
76. A vertical pole on a horizontal plane casts a shadow 50 ft. long when the angle of elevation (usually called *altitude*) of the sun equals 30°. Find the height of the pole.

77. From a cliff 300 ft. high, the angle of depression of a ship is 45°. Find the distance of the ship from the top of the cliff.



Ex. 77

78. A sailing vessel is propelled by the wind in an easterly direction at the rate of 10 mi. per hour. At the same time a current carries it N.E. at the rate of 8 mi. per hour. What is the true speed of the vessel per hour?



Ex. 78

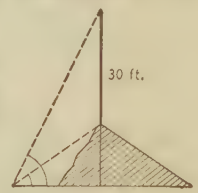
79. The angle of elevation of an inaccessible point P, measured at A, equals 45°. At a point B, 500 ft. more distant, the angle of elevation of P equals 22° 30′. If BAC is a horizontal line, what is the elevation of P above A (*i.e.* PC)?

Ex. 79

80. From a balloon which is directly above one town the angle of depression of another town is found to be 30°. Find the height of the balloon if the towns are 3 mi. apart.

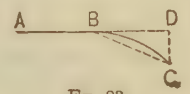
81. Two observers on opposite sides of a cloud measure the angles of elevation; $A = 45°$ and $B = 67° 30'$. Find the height of the cloud if $AB = 3$ mi.

82. A flagstaff 30 ft. high stands on top of a mound, and the angles of elevation of the top and the bottom of the pole are respectively 60° and 30°. Find the height of the mound.



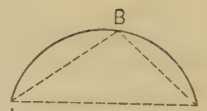
Ex. 82

83. The track ABC consists of a straight line AB and the arc BC. Find the radius of arc BC if line $BC = 50$ yd., and the perpendicular CD dropped upon the prolongation of AB equals 4 yd.



Ex. 83

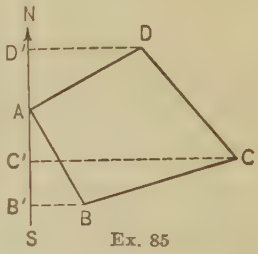
84. A circular track is to be constructed so as to pass through the points A, B, and C. If $AB = 50$ yd., $BC = 30$ yd., and $AC = 70$ yd., find the radius of the arc ABC.



Ex. 84

BOOK IV

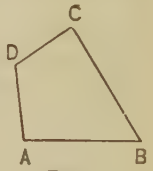
85. To measure the area of a quadrangular field $ABCD$, a surveyor measures the perpendiculars drawn to a line NS and the segments made on NS as follows: $AD' = 6$ rd., $AC' = 5$ rd., $C'B' = 4$ rd., $D'D = 7$ rd., $C'C = 12$ rd., and $B'B = 4$ rd. Find the area of $ABCD$.



Ex. 85

86. Determine the area of a pentagonal field $ABCDE$, if $AB = 18$, $BC = 24$, $CD = 28$, $DE = 24$, $EA = 10$, $AC = 30$, and $AD = 26$.

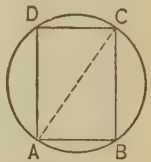
87. An irregular quadrangular field $ABCD$ must be divided into three equivalent parts by lines passing through A. Construct the lines of division.



Ex. 87

88. On a map whose scale is 1 : 500, how many square feet are represented by a polygon equivalent to 3 sq. in.?

89. From a given round log a rectangular beam is to be cut so as to make its cross section $ABCD$ as large as possible. Construct $ABCD$.



Ex. 89

90. A triangular field must be divided into 4 equivalent parts by lines parallel to one side. Construct the lines of division.

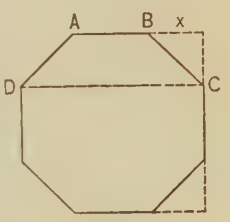
BOOK V

91. Construct the following patterns.

92. It is claimed that a rectangle (*e.g.* an envelope, a window, a picture, etc.) is most pleasing to the eye if its length and its width have the same ratio as the segments of a line which is divided in extreme and mean ratio. If a window is 4 ft. wide, what should be its height, if the two dimensions have the above-mentioned ratio?

93. The side of a tower whose base is a regular hexagon measures 10 ft. Find the area of the ground occupied by the tower.

94. AB, the side of the base of a regular octagonal tower, equals 8 ft. Find CD, the distance between two opposite walls.



Ex. 94

95. Find the area of the ground covered by the tower described in the preceding exercise.

96. The linoleum pattern in the figure consists of regular octagons (white) and squares (black). If the side of the squares equals 2 in., what is the length of AB, the distance between two opposite sides of an octagon?

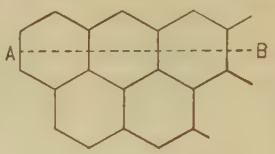
Ex. 96

97. In the same figure find the area of an octagon ($AD = 2$ in.).

98. If the length of AB is given, construct the pattern of the preceding exercise.

99. If the length of AB (preceding figure) equals 6 in., calculate the side of a square.

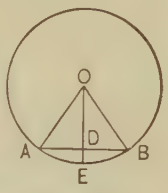
100. The floor of a room whose length is AB is covered with regular hexagonal tiles. How many tiles are in one row that extends over the entire length (AB) of the room, if each side of the hexagons equal 2 in., $AB = 10$ ft., and the tiles are placed in the position indicated in the figure?



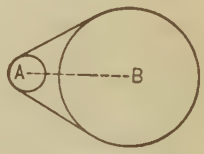
Ex. 100

101. The approximate value of the circumference of a circle is sometimes found by carpenters as follows: Draw the equilateral triangle AOB and produce its altitude OD to E. The circumference equals $6\,AO + 2\,DE$. Find the error if π is equal to 3.1416.

102. Two wheels A and B are connected by a belt, and their radii are respectively 2 ft. and 9 ft. If the larger wheel (B) makes 40 revolutions per minute, how many revolutions per minute will the smaller wheel make?



Ex. 101

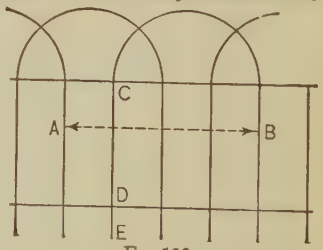


Ex. 102

103. In the figure of the preceding exercise find the length of the belt if $AB = 14$ ft. (Radii are 2 ft. and 9 ft.)

104. The gauge of an automobile (*i.e.* the distance between two wheels on the same axis) equals 4 ft. $8\frac{1}{2}$ in. How many inches more than an inner wheel does an outer wheel travel when the car turns around a rectangular corner?

105. The circumference of the equator is (approximately) 25,000 mi. Imagine a concentric circle whose circumference is 1 ft. longer. What is the difference of the radii of the two circles?



Ex. 106

106. The figure above represents a small wire fence used to protect flower beds, etc. How many feet of wire are needed per running foot of fence if $AB = 1$ ft., $CD = 9$ in., $DE = 3$ in.?

107. If two streets meet at an angle of 120°, and an automobile turns from one into the other, what is the difference between the distances traveled by an outer and an inner wheel of an automobile? (Gauge 4 ft. 8½ in.)

Ex. 108

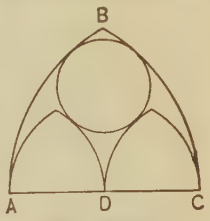
108. The span of a circular arch AB equals 10 ft., and its height $CD = 2$ ft. Find the radius of the arch.

109. If the earth's orbit be assumed to be a circle whose radius = 93,000,000 mi., and the year = 365 da., how many miles per hour does the earth travel?

110. Find the area of an equilateral Gothic arch ABC, if $AC = 4$ ft.

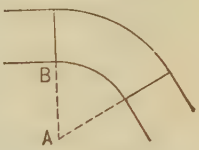
111. In the same figure, what is the area of each of the smaller equilateral arches if $AD = DC = 2$ ft.?

112. In the same diagram, what is the area of the circle that touches the three equilateral arches?



Exs. 110–112

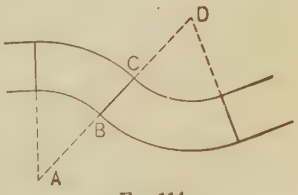
113. A sidewalk 5 ft. wide turns a corner. Find the area of the curved portion of the sidewalk if the radius $AB = 8$ ft. and $\angle A = 60°$.



Ex. 113

114. Determine the area of the curved portion of the road drawn in the figure, if $AB = 90$ ft., $BC = 50$ ft., $CD = 90$ ft., $\angle A = 45°$, and $\angle D = 60°$.

115. How many miles per hour does a point whose latitude equals 45° travel in consequence of the earth's rotation about its axis P if we assume the diameter of the earth equal to 8000 mi., and the time of one rotation equal to 23 hr. 56 min.?

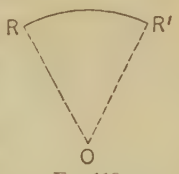


Ex. 114

116. It is found that a water pipe whose diameter is 2 in. supplies half the amount of water that is needed in a building. What would be the diameter of a pipe that would supply the entire required amount of water?

117. What should be the diameter of a water pipe that supplies the same amount of water as both an 8-in. and a 6-in. pipe, if we assume that all other conditions, as pressure, etc., are the same in the three pipes?

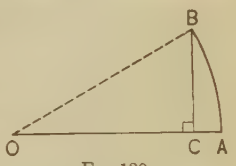
118. A rail 50 ft. long must be bent through what angle (*i.e.* $\angle O$ formed by perpendiculars at its end) if the radius of curvature equals 360 ft.?



Ex. 118

119. In laying a curved track, a rail 55 ft. long is bent through an angle of 17° 30′. Find the radius of curvature.

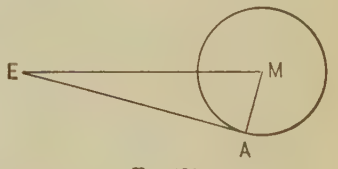
120. If O is the center of arc AB, $BC \perp OA$, and $\angle O$ is small, then the difference between $\overset{\frown}{AB}$ and BC is very small. If we have to find the value of BC, we may, in such cases, find the length of arc AB instead. If $\angle O = 4°$, the error due to this simplification equals .00005 of the result, and for smaller angles the error is much smaller, since it is approximately proportional to the cube of the angle.



Ex. 120

If BC is a tower standing on a horizontal plane AO, $BO = 2000$ ft. and $\angle O = 2°$. Find the height of the tower.

121. The apparent diameter of the moon equals 30′, *i.e.* a line EM drawn from earth (E) to the center of the moon (M) and a tangent EA include an angle of 15′. If $EM = 243{,}000$ mi., find the radius of the moon.



Ex. 121

122. The apparent diameter of the sun is approximately the same as that of the moon (preceding problem). What is the diameter of the sun if its distance from the earth equals 93,000,000 mi.?

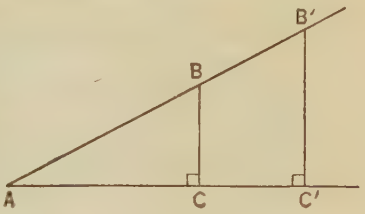
123. Applying Trigonometric Functions (pp. 301–304), solve problems 75–82.

124. By the same method, solve problems designated in 123, first replacing $\angle 30°$ by $\angle 33°$, $\angle 60°$ by $\angle 64°$, and $\angle 45°$ by $\angle 40°$.

TRIGONOMETRIC FUNCTIONS

465. In order to be able to solve many problems involving angles different from 60° and 90° and their multiples and submultiples, we have to turn to the trigonometric functions of angles. A trigonometric function of an angle is a ratio that depends upon the angle for its value.

Let $\angle A$ be given and BC a fixed line drawn from a point in one side of the angle perpendicular to the other side ($BC \perp AC$). From any other point as B' draw the line $B'C' \perp AC$.



Then right $\triangle ABC \sim$ right $\triangle AB'C'$. (Why?)

$$BC : B'C' = AB : AB' \qquad (303)$$

and
$$BC : AB = B'C' : AB' \qquad (279)$$

or
$$\frac{BC}{AB} = \frac{B'C'}{AB'}.$$

But $\frac{BC}{AB}$ is constant for a fixed $\angle A$. (Why?)

Hence $\frac{B'C'}{A'B'}$ = the same constant for every position of B'. (Why?)

466. This ratio is called the sine of angle A.

Likewise the ratio $\frac{AC}{AB}$ is called the cosine of angle B.

Also the ratio $\frac{BC}{AC}$ is called the tangent of angle A.

The numerical values of these ratios have been found by more advanced mathematics and placed in tables convenient for our use. Acute angles only are considered in this textbook.

467. DEF. The **sine** of an angle equals $\frac{\text{side opp. the angle}}{\text{hypotenuse}}$.

The **cosine** of an angle equals $\frac{\text{side adjacent to the angle}}{\text{hypotenuse}}$.

The **tangent** of an angle equals $\frac{\text{side opp. the angle}}{\text{side adj. to the angle}}$.

Using the table (p. 304), work out the following exercises:

Ex. 1. What angle has its sine equal to .326? To .500? To .946? To .993? To .755?

Ex. 2. What angle has its cosine equal to .970? To .707? To .309? To .883? To. 438?

Ex. 3. What angle has its tangent equal to .158? To 2.145? To 5.671? To .625? To .249?

Ex. 4. Find, to the nearest degree, the angle having its sine equal to .450.

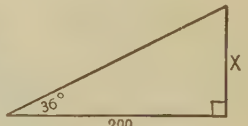
Ex. 5. Find, to the nearest degree, the angle having its cosine equal to .450.

Ex. 6. Find, to the nearest degree, the angle having its tangent equal to .290.

Ex. 7. From the table, as the angle increases, how do its three functions behave?

Ex. 8. The angle of elevation (see p. 287, Ex. 34) of the top of a tower 200 feet from its base is 36°. Find the height of the tower.

HINT. $\frac{x}{200} = \tan 36°$.



Ex. 8

Ex. 9. The angle of depression (see p. 287, Ex. 34) of a boat from the top of a lighthouse 84 feet high is 17°. How far is the boat from the foot of the lighthouse? HINT. $\tan 17° = \frac{84}{d\text{(istance)}}$.

Ex. 10. A balloon is held to the ground by a taut wire rope that is 328 feet long. If the rope makes an angle of 67° with the ground, how high is the balloon?

Ex. 11. In the following problems, the right triangle has its right angle at C. a and b are the sides opposite angles A and B respectively, and c is the hypotenuse. Substitute for the question marks.

$\angle A$	$\angle B$	a	b	c
48°	?	?	?	280
?	37°	14.2	?	?
?	?	8.3	10.2	?
?	?	?	6.5	12.4
71°	?	$3\frac{1}{3}$	?	?

Ex. 12. Find the area of a right triangle given the hypotenuse 18 and one acute $\angle 40°$.

Ex. 13. How many square feet are there in the area of a triangle having two sides and their included angle respectively 41 feet, 36 feet, and $\angle 38°$?

Ex. 14. From a point C on one bank of a river, exactly opposite a tree B on the other bank, I walk at right angles to CB to a point A so that CA equals 120 yards. I find the angle CAB is 28°. What is the width of the river?

Ex. 15. From a point P two tangents are drawn to a circle. These tangents are each 45 feet long and form an angle of 68°. Find the diameter of the circle.

Ex. 16. The base angles of an isosceles triangle are each 34°. If the base is 80 ft., what is the altitude of the triangle?

Ex. 17. The angle formed by the two radii drawn to the extremities of a chord is 130°. If the radii are 11 feet, how long is the chord?

Ex. 18. The altitude on the hypotenuse of a right triangle is 14 feet. One angle of the triangle is 24°. Find the length of the hypotenuse.

Ex. 19. Solve problems 38–42 in Practical Applications (pages 288–289).

Ex. 20. Solve problems 123, 124, on page 300.

VALUES OF TRIGONOMETRIC FUNCTIONS

Degree	Sine ∠	Cos ∠	Tan ∠	Degree	Sine ∠	Cos ∠	Tan ∠
0	.000	1.000	.000	**45**	.707	.707	1.000
1	.017	.999	.017	**46**	.719	.695	1.036
2	.035	.999	.035	**47**	.731	.682	1.072
3	.052	.998	.052	**48**	.743	.669	1.111
4	.070	.997	.070	**49**	.755	.656	1.150
5	.087	.996	.087	**50**	.766	.643	1.192
6	.105	.995	.105	**51**	.777	.629	1.235
7	.122	.993	.123	**52**	.788	.616	1.280
8	.139	.990	.140	**53**	.799	.602	1.327
9	.156	.988	.158	**54**	.809	.588	1.376
10	.174	.985	.176	**55**	.819	.574	1.428
11	.191	.982	.194	**56**	.829	.559	1.483
12	.208	.978	.213	**57**	.839	.545	1.540
13	.225	.974	.231	**58**	.848	.530	1.600
14	.242	.970	.249	**59**	.857	.515	1.664
15	.259	.966	.270	**60**	.866	.500	1.732
16	.276	.961	.287	**61**	.875	.485	1.804
17	.292	.956	.306	**62**	.883	.469	1.881
18	.309	.951	.325	**63**	.891	.454	1.963
19	.326	.946	.344	**64**	.899	.438	2.050
20	.342	.940	.364	**65**	.906	.423	2.145
21	.358	.934	.384	**66**	.914	.407	2.246
22	.375	.927	.404	**67**	.921	.391	2.356
23	.391	.921	.424	**68**	.927	.375	2.475
24	.407	.914	.445	**69**	.934	.358	2.605
25	.423	.906	.466	**70**	.940	.342	2.748
26	.438	.899	.488	**71**	.946	.326	2.904
27	.454	.891	.510	**72**	.951	.309	3.078
28	.469	.883	.532	**73**	.956	.292	3.271
29	.485	.875	.554	**74**	.961	.276	3.487
30	.500	.866	.577	**75**	.966	.259	3.732
31	.515	.857	.601	**76**	.970	.242	4.011
32	.530	.848	.625	**77**	.974	.225	4.332
33	.545	.839	.649	**78**	.978	.208	4.705
34	.559	.829	.675	**79**	.982	.191	5.145
35	.574	.819	.700	**80**	.985	.174	5.671
36	.588	.809	.727	**81**	.988	.156	6.314
37	.602	.799	.754	**82**	.990	.139	7.115
38	.616	.788	.781	**83**	.993	.122	8.144
39	.629	.777	.810	**84**	.995	.105	9.514
40	.643	.766	.839	**85**	.996	.087	11.430
41	.656	.755	.869	**86**	.997	.070	14.300
42	.669	.743	.900	**87**	.998	.052	19.081
43	.682	.731	.933	**88**	.999	.035	28.636
44	.695	.719	.966	**89**	.999	.017	57.290

A SKETCH OF THE HISTORY OF GEOMETRY

468. Of course, the history of geometry cannot be traced actually to its beginning. No records were made at first and, no doubt, most of the earliest records are entirely lost to the world. Painstaking investigations make probable most of the statements that follow here.

469. The ancient **Babylonians** and **Chaldeans**, on account of their religious observances, made some study of geometry. Most of their knowledge was got merely through experiences in a practical way.

470. It is, however, to the ancient **Egyptians** that we owe the beginnings of geometry that spread through the countries of Europe, Western Asia, and the Americas. The necessity of building pyramids and temples, of the re-division of land after an overflow of the Nile, caused their engineers to collect empirical rules, which they applied in their various labors.

The oldest known manuscript dates back possibly to 1700 B.C. It was written by **Ahmes**, who says it was based upon an earlier work. Among other mathematical ideas, it contains rules on mensuration.

These Egyptians knew how to find the area of a rectangle exactly, but they thought the area of an isosceles triangle could be found by taking one half its base times one of its legs. They used 3.16 as a value for π. They knew how to lay off a right angle by using a right triangle having its three sides in the ratio of $3:4:5$. Of the logical foundations of geometry, they knew very little indeed.

471. The ancient **Chinese** and **Japanese** probably assumed π to equal either 3 or $\sqrt{10}$. They had a few simple rules for mensuration based upon observation.

Likewise the ancient **Hindus** knew but little of geometry, most of which they probably got in the same manner as other

ancient races. In later times the Hindus imported their ideas from the Greeks and, although they used the elementary principles of geometry, they gave no real logical proofs.

472. For the beginnings of demonstrative geometry we are indebted to the spirit of Greek investigation. The **Greeks** studied in Egypt, became acquainted with their simple rules in their applications, but went further and gave to the world the logical principles underlying elementary geometry.

To **Thales** (c. 575 B.C.), although he left no written records, is attributed the introduction of geometry into Greece. He knew that (1) vertical angles are equal, (2) the base angles of an isosceles triangle are equal, (3) two triangles are congruent if they have a side in each equal and the angles at the extremities of these sides respectively equal, (4) the diameters of a circle bisect the circle, (5) and, possibly, triangles that are mutually equiangular are similar.

He was the first Greek known to apply geometry to practical problems.

To **Pythagoras** (c. 540 B.C.) is ascribed the Pythagorean theorem: The square on the hypotenuse equals the sum of the squares on the other two sides. However, the earliest proof of this theorem that has come down to us is to be found in **Euclid.** The School of Pythagoras made geometry the foundation of their teachings. Of the various theorems that they proved may be mentioned: (1) the sum of the angles of a triangle equals two right angles, (2) to construct a parallelogram equivalent to a triangle, and (3) the diagonal of a square is incommensurable with a side. They did not add anything to the knowledge of the circle.

This school proved the various theorems by the *deductive* method.

The **Sophists** studied the circle and attempted to solve the so-called three famous problems of antiquity: (1) the

trisection of any angle, (2) the duplication of the cube, and (3) the squaring of a circle. Although they failed completely in using elementary methods, they were led to discover many properties of circles.

Hippocrates (c. 460 B.C.) contributed much to the geometry of circles, applied logic in his proofs, and added theorems on similar figures and proportion. He wrote a textbook called the *Elements*.

Plato (c. 380 B.C.) and his school trained men's minds for correct and vigorous thinking, demanded that the straight edge and compasses be the only instruments for constructions, and introduced *analysis* as a method to be used in geometrical proofs.

Aristotle (c. 340 B.C.), although not much interested in mathematics as such, gave principles for forming logical definitions.

Euclid (c. 300 B.C.) is the author of the greatest textbook in the world. Other textbooks had been written on geometry, but it remained for Euclid to systematize the work of his predecessors, to remove imperfections, and to put on a firm logical basis the propositions in elementary geometry.

Archimedes (c. 225 B.C.) was the greatest mathematical physicist of antiquity. In his works we find the beginnings of much of the modern mathematics. He found that π must lie within the limits of $3\frac{1}{7}$ and $3\frac{10}{71}$.

473. The **Romans** carried on almost no independent study of theoretical geometry. What they obtained from the Greeks they applied to their practical problems in engineering. They even used $3\frac{1}{8}$ as a value for π, since it fitted in with their duodecimal system of notation of numbers.

474. The **Arabs** added little to the geometry known in their day. However, by translating many textbooks into their language and thereafter transmitting into Europe the

Greek geometry, they did a signal service for mankind. In a later period the Arabs reduced many geometric problems to the solution of algebraic equations. This helped to unify mathematics.

475. During the **Middle Ages** Euclid held sway, and it was not until **Legendre**, a Frenchman, in 1794 wrote his *Elements of Geometry*, that the world began to use textbooks in geometry simply enough written to be understood by the students of lower schools. To-day we find many new kinds of geometry; such as non-Euclidean, projective, descriptive, hyper-space, and analytic.

476. It is hoped that many students may become sufficiently interested through this sketch to desire to read more about the history of mathematics. Excellent material will be found in Cajori's *A History of Mathematics*.

NUMERICAL COMPUTATION

477. Approximation to nearest hundredth.

Ex. Find to nearest hundredth the values of $\frac{1}{8}$, $1\frac{2}{3}$, .63755, $\frac{9.217}{4}$.

$\frac{1}{8}$ = .125 and, to the nearest hundredth, is .13.
$1\frac{2}{3}$ = 1.666 and, to the nearest hundredth, is 1.67.
.63755 to the nearest hundredth is .64.
$\frac{9.217}{4}$ = 2.304 and, to the nearest hundredth, is 2.30.

478. The square roots of numbers.

Ex.
$$\begin{array}{r} 4.97- \\ \sqrt{24.7560} \\ 16 \\ \hline 89\,|\,8\,75 \\ 8\,01 \\ \hline 987\,|\,6460 \\ 6909 \end{array}$$

Ans. 4.97 to nearest hundredth.

Simplification of radicals.

Ex. Rationalize $\frac{2}{\sqrt{3}}$ and find approximate value to the nearest hundredth.

$$\frac{2}{\sqrt{3}} = \frac{2}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{3} = \frac{2(1.732)}{3} = \frac{3.464}{3} = 1.154^{+}.$$

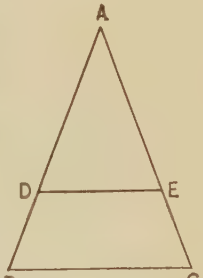
Its value to nearest hundredth is 1.15.

479. The equation in geometry.

Ex. 1. Find the angle whose supplement is 32° more than two and one half times its complement.

Sol.
$$180 - x = 2\tfrac{1}{2}(90 - x) + 32.$$
$$180 - x = \tfrac{5}{2}(90 - x) + 32.$$
$$360 - 2x = 450 - 5x + 64.$$
$$3x = 154.$$
$$x = 51\tfrac{1}{3}° \text{ or } 51° 20'.$$

Ex. 2. Given $AB = 10$, $AE = 8$, $EC = 5$, and $DE \parallel BC$, to find AD and DB in $\triangle ABC$.



Ex. 2

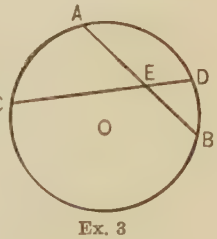
Sol. Let $x = AD$, then $10 - x = DB$,

but $AD : DB = AE : EC$

or $x : 10 - x = 8 : 5.$

$$5x = 8(10 - x)$$
$$5x = 80 - 8x.$$
$$13x = 80.$$
$$x = 6\tfrac{2}{13} = AD.$$
$$10 - x = 3\tfrac{11}{13} = DB.$$

Ex. 3. In circle O, CD is cut by AB so that $CE = 12$ and $ED = 2$. If $AB = 11$, find segments AE and EB.



Ex. 3

Sol. $CE \cdot ED = AE \cdot EB.$

Let $x = AE$, then $11 - x = EB.$

Then $12 \times 2 = x(11 - x).$

$$24 = 11x - x^2.$$
$$x^2 - 11x = -24.$$
$$x^2 - 11x + (\tfrac{11}{2})^2 = \tfrac{121}{4} - 24 = \tfrac{121}{4} - \tfrac{96}{4} = \tfrac{25}{4}.$$
$$(x - \tfrac{11}{2})^2 = \tfrac{25}{4}.$$
$$x - \tfrac{11}{2} = \sqrt{\tfrac{25}{4}} = \pm \tfrac{5}{2}.$$
$$x = \pm \tfrac{5}{2} + \tfrac{11}{2} = 8 \text{ or } 3.$$
$$11 - x = \quad 3 \text{ or } 8.$$

$\therefore AE = 8$ and $EB = 3$, or $AE = 3$ and $EB = 8$.

Ex. 4. Find the radius of a circle having its circumference equal to the sum of three circumferences of circles with radii 3, 4, and 5 respectively.

Sol. Let r = radius of required circle.

Then $2\pi r = 2\pi(3) + 2\pi(4) + 2\pi(5)$,

and $r = 3 + 4 + 5 = 12$.

Ex. 5. Given a circle of radius 4 in., to find a side of an equivalent equilateral triangle.

Sol. Let a = side of required triangle.

Then $\frac{a^2}{4}\sqrt{3} = \pi r^2 = 16\pi$.

$$a^2 = \frac{64\pi}{\sqrt{3}}.$$

$$a^2 = \frac{64\pi\sqrt{3}}{3} = \frac{64(3.1416)(1.732)}{3}.$$

$$a^2 = 116.08$$

$$a = \sqrt{116.08} = 10.8^- \text{ in.}$$

TABLE OF THE MOST IMPORTANT FORMULAS

NOTATION

a, b, c = sides of $\triangle ABC$.

m_a = median to a.

h_a = altitude to a.

$s = \frac{1}{2}(a + b + c)$.

b = base.

c = circumference.

s_n = side of regular polygon of n sides.

b' = projection of b upon c.

R = radius of circumscribed $\odot$.

t_c = bisector of angle opp. c.

r = radius of inscribed $\odot$, or apothem.

S = area (surface units).

d = diameter; also diagonal of a square or rhombus.

b_1 and b_2 = bases of a trapezoid.

p = perimeter.

FORMULAS FOR LINE VALUES

Right $\triangle$, $a^2 + b^2 = c^2$ (c = hy.). (329)

Oblique $\triangle$, $a^2 = b^2 + c^2 \pm 2\,b'c$, dependent upon $\angle A$ being obtuse or acute. (334, 336)

Circle, $c = 2\,\pi R = \pi\,d$. (424)

Diagonal of a square, $d = b\sqrt{2}$.

Equilateral $\triangle$, $h = \frac{b}{2}\sqrt{3}$. (p. 189, Ex. 4.)

$s_3 = R\sqrt{3}$. (401)

$s_4 = R\sqrt{2}$. (396)

$s_6 = R$. (398)

$h_c = \frac{2}{c}\sqrt{s(s-a)(s-b)(s-c)}$. (339)

$m_c = \frac{1}{2}\sqrt{2(a^2+b^2) - c^2}$. (340, Ex. 5)

$t_c = \frac{2}{a+b}\sqrt{abs(s-c)}$. (341, ex. 6)

In similar polygons:

$p : p' = a : a'$ (318)

and $a : a' = b : b' = c : c'$, etc. (303)

$s_{2n} = \sqrt{2\,R^2 - R\sqrt{4\,R^2 - s_n^2}}$. (432)

An arc $= \frac{\text{central } \angle}{360^\circ} \times 2\,\pi R$. (425)

In circles,

$c : c' = d : d' = r : r'$. (421, 422)

AREAS OF PLANE FIGURES

Rectangle,	bh.	(350)
Square,	b^2.	(351)
Parallelogram,	bh.	(353)
Triangle,	$\frac{1}{2}\,bh$, or $\sqrt{s(s-a)(s-b)(s-c)}$.	(358)
Equilateral $\triangle$,	$\frac{b^2}{4}\sqrt{3}$.	(p. 228, Ex. 12)
Trapezoid,	$\frac{1}{2}\,h(b_1 + b_2)$.	(363)
Rhombus,	$\frac{1}{2}\,d_1\,d_2$.	(p. 228, Ex. 13)
Regular polygon,	$\frac{1}{2}\,pr$.	(426)
Circle,	πR^2 or $\frac{1}{2}\,Rc$.	(427, 428)
Sector of circle,	$\frac{1}{2}$ arc $\times\, R$ or $\frac{\text{central } \angle}{360} \times \pi R^2$.	(430)

Segment of $\odot$, Sector $\pm$ triangle formed by chord and radii drawn to the extremities of the chord.

Similar polygons,	$S : S' = a^2 : a'^2 = p^2 : p'^2$.	(380, 382)
Two circles,	$S : S' = R^2 : R'^2 = d^2 : d'^2$.	(429)

TABLES

Linear Measure

12 inches (in.) = 1 foot (ft.)
3 feet = 1 yard (yd.)
5½ yards = 1 rod
320 rods = 1 mile (mi.)
1 mile = 1760 yd. = 5280 ft.

Square Measure

144 square in. = 1 sq. ft.
9 sq. ft. = 1 sq. yd.
30¼ sq. yd. = 1 sq. rod
160 sq. rods = 1 acre
640 acres = 1 sq. mi.

METRIC SYSTEM

Linear Measure

10 millimeters (mm.) = 1 centimeter (cm.)
10 centimeters = 1 decimeter (dm.)
10 decimeters = 1 meter (m.)
10 meters = 1 dekameter (dm.)
10 dekameters = 1 hectometer (hm.)
10 hectometers = 1 kilometer (km.)
1 meter = 39.37 in.

Square Measure

100 sq. mm. = 1 sq. cm.
100 sq. cm. = 1 sq. dm.
100 sq. dm. = 1 sq. meter

Approximations

$\sqrt{2} = 1.414$ $\qquad$ $\pi = 3.1416$
$\sqrt{3} = 1.732$ $\qquad$ $\pi = 3.14$
$\sqrt{5} = 2.236$ $\qquad$ $\pi = 3\frac{1}{7}$

SOLID GEOMETRY

BOOK VI

LINES AND PLANES IN SPACE — POLYHEDRAL ANGLES

480. DEF. **Geometry of space** or **solid geometry** treats of figures whose elements are not all in the same plane.

481. DEF. A **plane** is a surface such that a straight line joining any two points in it lies entirely in the surface.

A **plane is determined** by given points or lines, if one plane, and only one, can be drawn through these points or lines.

Post. A. One and only one plane can be passed through three points not in the same straight line.

Post. B. If two planes have one point in common, they have a second point in common.

482. DEF. A **straight line and a plane are parallel** if they do not meet, however far they may be produced.

483. DEF. **Two planes are parallel** if they do not meet, however far they may be produced.

NOTE. Constructions in solid geometry are usually treated as if we possessed the physical means of *making* a plane that is determined by any of the above elements. Obviously this could be done only if we employed models. With ruler and compasses the work is largely symbolic; *i.e.* we obtain pictures which are symbols of the figures to be constructed, but we do not always obtain the required figures themselves, as we do in plane geometry.

PROPOSITION I. THEOREM

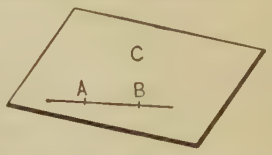
484. *A plane is determined:*

(1) *By a straight line and a point without the line.*

(2) *By two intersecting straight lines.*

(3) *By two parallel straight lines.*

(1) To prove that a plane is determined by a straight line AB and a point C outside AB.



Through two points A, B on the line, and point C one, and only one, plane can be passed. (481, Post. A)

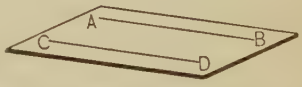
Line AB lies in this plane. (481)

(2) To prove that two intersecting straight lines AB and AC determine a plane.

[Proof by the student.]

(3) To prove that two parallel lines AB and CD determine a plane.

The parallel lines AB and CD lie in the same plane by definition.



Since AB and the point C determine a plane, the two parallels determine a plane.

485. COR. *The intersection of two planes is a straight line.*

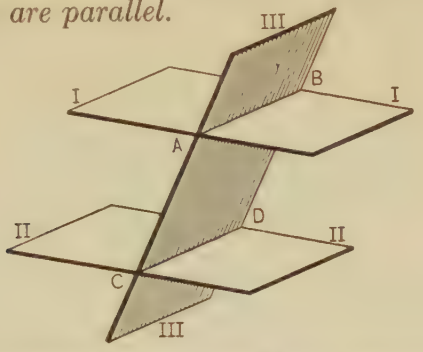
For the intersection cannot contain three points not in a straight line, since only one plane can be passed through three such points. (483, Post. B)

Ex. 1. Why is a photographer's camera or a surveyor's transit supported by three legs?

Ex. 2. How many planes are determined by four points which do not all lie in one plane?

PROPOSITION II. THEOREM

486. *The intersections of two parallel planes with a third plane are parallel.*



Given the parallel planes I and II, intersected by a third plane III in AB and CD, respectively.

To prove $AB \parallel CD$.

Proof. AB and CD cannot meet, for otherwise planes I and II would meet.

AB and CD lie in the same plane.

Hence $AB \parallel CD$.

487. COR. *Parallel lines included between parallel planes are equal.*

EX. 1. *If a plane intersects one of two parallel planes, it intersects the other plane.*

EX. 2. *If a line intersects one of two parallel planes, it intersects the other plane.*

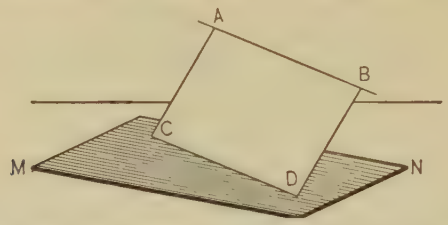
EX. 3. Point out in the schoolroom some illustrations of Prop. II.

EX. 4. In the fig. for Prop. II, if $AC \parallel BD$, show that $AB = CD$.

NOTE. The student should note that in order to prove the parallelism of two lines in solid geometry it is not sufficient to show that the lines cannot meet. It must be demonstrated also that the two lines lie in one plane.

Proposition III. Theorem

488. *A plane containing one, and only one, of two parallel lines is parallel to the other line.*



Given $AB \parallel CD$, and plane MN containing CD only.

To prove plane $MN \parallel AB$.

Proof. AB and CD determine a plane intersecting MN in CD.

Hence, if AB meets MN, it must meet MN in CD.

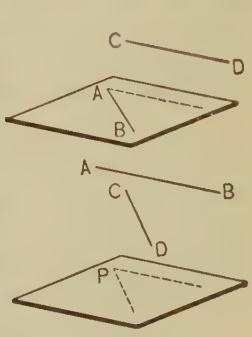
But, since $AB \parallel CD$, this is impossible.

Hence plane $MN \parallel AB$.

489. Remark. Proposition III furnishes a method for constructing a plane parallel to a given line, AB. In such a construction always construct first a line, CD, parallel to the given line AB, and then pass a plane through CD.

490. Cor. 1. *Through a given straight line a plane can be passed parallel to any other given straight line; and if the lines are not parallel, only one such plane can be drawn.*

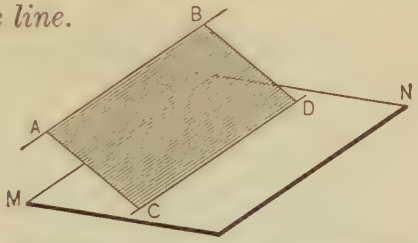
491. Cor. 2. *Through a given point a plane can be passed parallel to any two given straight lines in space; and if the lines are not parallel, only one such plane can be drawn.*



Ex. To construct a plane parallel to a given line and passing through two given points.

PROPOSITION IV. THEOREM

492. *A line parallel to a plane is parallel to the intersection of this plane with any plane passing through the line.*



Given $AB \parallel$ plane MN, and plane BC containing AB and intersecting MN in CD.

To prove $AB \parallel CD$.

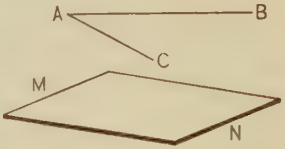
Proof. AB and CD cannot meet, for otherwise AB would meet the plane MN.

AB and CD are in the same plane.

Hence $AB \parallel CD$.

493. COR. 1. *If each of two intersecting lines is parallel to a given plane, their plane is parallel to the given plane.*

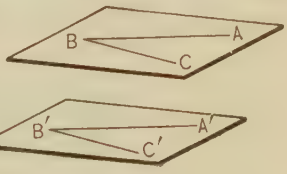
Let both AB and AC be parallel to MN. If plane ABC should intersect MN, the intersection would be parallel to both AB and AC, which is impossible.



494. COR. 2. *If two angles have their sides respectively parallel, their planes are parallel.*

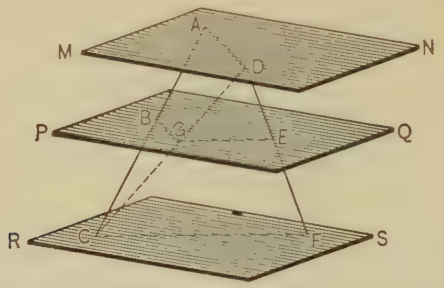
Since $AB \parallel A'B'$, plane ABC is parallel to $A'B'$; and similarly plane ABC is parallel to $B'C'$.

Hence plane $ABC \parallel$ plane $A'B'C'$. (Cor. 1.)



Proposition V. Theorem

495. *If two straight lines are cut by parallel planes, their corresponding segments are proportional.*



Given parallel planes MN, PQ, and RS intersected by two straight lines in A, B, C and D, E, F, respectively.

To prove $AB : BC = DE : EF$.

Proof. Draw DC, and pass a plane through AC and DC, intersecting the planes PQ and MN in BG and AD, respectively.

Then $AD \parallel BG$. (486)

By passing a plane through DC and DF, it follows, similarly,

$GE \parallel CF$.

Hence $\frac{AB}{BC} = \frac{DG}{GC}$, and $\frac{DE}{EF} = \frac{DG}{GC}$. (Why?)

$\therefore \frac{AB}{BC} = \frac{DE}{EF}$. (Ax. 1.)

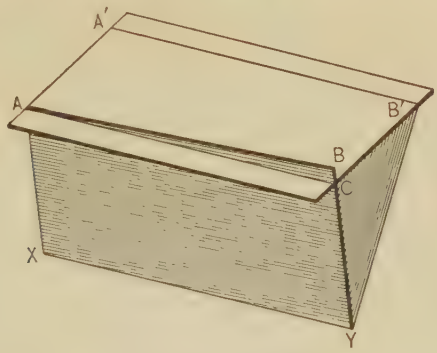
496. Cor. *If through any point straight lines are drawn intersected by two parallel planes, their corresponding segments are proportional.*

Ex. **1.** *If three parallel planes intercept equal parts on one transversal, they intercept equal parts on every transversal.*

Ex. **2.** If in the diagram for Prop. V, $BG = 5$, $AD = 15$, $DE = 4$, find EF.

PROPOSITION VI. THEOREM

497. *If two straight lines are parallel to a third straight line, they are parallel to each other.*



Given $AB \parallel XY$, and $A'B' \parallel XY$.

To prove $AB \parallel A'B'$.

Proof. Pass plane AY through AB and XY, and plane $B'A$ through $A'B'$ and point A.

Let planes AY and $B'A$ intersect in AC.

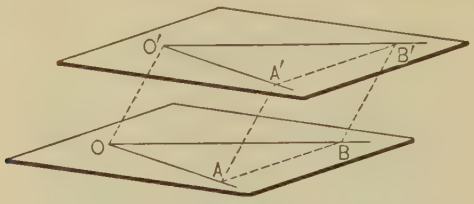
	Plane $B'A \parallel XY$.	(488)
Hence	$AC \parallel XY$.	(492)
But	$AB \parallel XY$.	(Hyp.)
Therefore	AB and AC coincide.	(Ax. 16.)
	AB and $A'B'$ lie in one plane.	
But	AB and $A'B'$ cannot meet.	(Ax. 16.)
Hence	$AB \parallel A'B'$.	

EX. If a line AB is parallel to a plane P, and parallel to another line CD, then plane $P \parallel CD$.

HINT. Pass a plane through AB intersecting P in a line XY.

PROPOSITION VII. THEOREM

498. *If two angles, not in the same plane, have their sides parallel and lying in the same direction, they are equal.*



Given the sides of the angles AOB and $A'O'B'$ respectively parallel and lying in the same direction.

To prove $\angle AOB = \angle A'O'B'$.

Proof. Take $OA = O'A'$, and $OB = O'B'$.

Draw AA', BB', AB, and $A'B'$.

AO' is a parallelogram. (Why?)

BO' is a parallelogram. (Why?)

Hence AA' is equal and parallel to OO',

and BB' is equal and parallel to OO'.

Therefore AA' is equal and parallel to BB'. (Ax. 1.) (497)

Whence $AA'B'B$ is a parallelogram,

and $AB = A'B'$.

$\therefore \triangle AOB \cong \triangle A'O'B'$. (Why?)

$\therefore \angle O = \angle O'$.

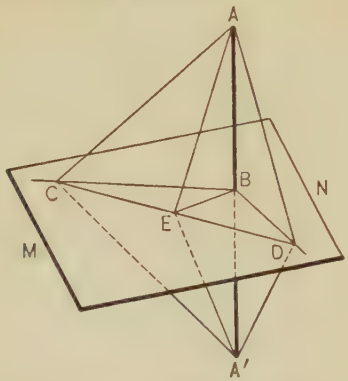
499. DEF. The **foot** of a line intersecting a plane is the point of intersection.

500. DEF. A **straight line is perpendicular to a plane** if it is perpendicular to every line **drawn through its foot** in the plane.

501. DEF. A **plane is perpendicular to a line** if the line is perpendicular to the plane.

PROPOSITION VIII. THEOREM

502. *If a straight line is perpendicular to each of two lines at their point of intersection, it is perpendicular to the plane of these lines.*



Given BC and $BD \perp AB$, and plane MN containing BC and BD.

To prove $AB \perp$ plane MN.

Proof. In plane MN draw any line BE through B.

Draw CD meeting BE in E and produce AB to A', so that $BA' = AB$.

Draw AC, AE, AD, CA', EA', and DA'.

$AC = A'C$, $AD = DA'$. (125)

$CD = CD$. (Why?)

Hence $\triangle ACD \cong \triangle CDA'$. (Why?)

$\therefore \angle ACD = \angle A'CD$,

and $\triangle ACE \cong \triangle A'CE$. (Why?)

$\therefore EA = EA'$.

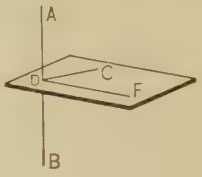
Then, $BE \perp AA'$. (79)

Hence, AB is perpendicular to any line passing through its foot in the plane MN, and $AB \perp$ plane MN. (500)

503. Cor. *Through a point to pass a plane perpendicular to a line.*

I. Given line AB and point D in line AB. Pass two planes through AB. In these planes (not in figure) draw $DC \perp AB$ and $DF \perp AB$. Plane CDF is the required plane.

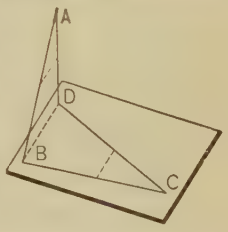
II. Given line AB and point C without the line. Draw $CD \perp AB$. Pass a plane through AB not containing point C. In this plane draw $DF \perp AB$. The required plane is CDF.



Ex. **1**. In the diagram for Prop. VIII, find the length of CD, if $AD = 5$, $AB = 4$, $BC = 5$, $\angle CBD = 120°$, and $AB \perp$ plane MN.

Ex. **2**. If $ABCD$ is a quadrilateral in space (*i.e.* A, B, C, and D do not all lie in the same plane) and $AB = BC$, $CD = DA$, then plane angle A is equal to plane angle C.

Ex. **3**. A line joining the mid-points of two adjacent sides of a quadrilateral in space is equal and parallel to a line joining the mid-points of the other two sides.



Ex. **4**. If two angles, not in the same plane, have their sides parallel and lying in opposite directions, they are equal.

Ex. **5**. Under what condition will two angles be supplementary, if their sides are parallel?

Ex. **6**. Under what condition can a plane be passed through two points and be perpendicular to a given line?

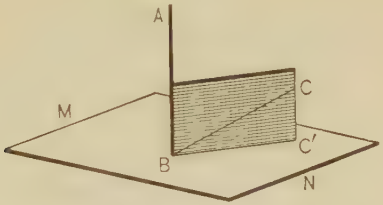
Ex. **7**. How many planes can be passed through five points, no four being in a same plane?

Ex. **8**. What is the least number of planes that can inclose a definite portion of space? Why?

Ex. **9**. Prove: (*a*) A quadrilateral determines a plane if its two diagonals intersect. (*b*) If two of its sides are parallel. (*c*) If two of its opposite sides intersect.

PROPOSITION IX. THEOREM

504. *All the perpendiculars that can be drawn to a straight line at a given point lie in a plane perpendicular to the line at that point.*



Given $AB \perp BC$, and $AB \perp$ plane MN at B.

To prove BC lies in plane MN.

Proof. Pass a plane through AB and BC, and let it intersect MN in BC'.

$AB \perp BC'$. (500)

$AB \perp BC$. (Hyp.)

Hence BC and BC' coincide, (47)

and BC lies in plane MN.

505. COR. 1. *At a given point in a straight line, one, and only one, plane can be drawn perpendicular to the straight line.*

506. COR. 2. *Through a point without a straight line, one plane, and only one, can be drawn perpendicular to that line.*

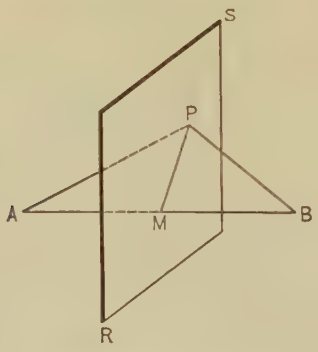
Ex. **1**. A right angle is revolved about one of its sides as an axis. What kind of surface does the moving side generate?

Ex. **2**. Three lines are mutually perpendicular to each other. Prove that no fourth line can be drawn perpendicular to all three given lines.

Ex. **3**. Prove that no line can be constructed perpendicular to two planes having a point in common.

Proposition X. Theorem

507. *Every point in the plane that bisects a line at right angles is equidistant from the ends of the line.*



Given point P in plane RS which is $\perp$ to line AB at its mid-point M.

To prove every point P is equidistant from A and B; *i.e.*

$$PA = BP.$$

Proof. PM is a $\perp$ bisector of AB. (Why?)

$$PA = PB \quad \text{(Why?)}$$

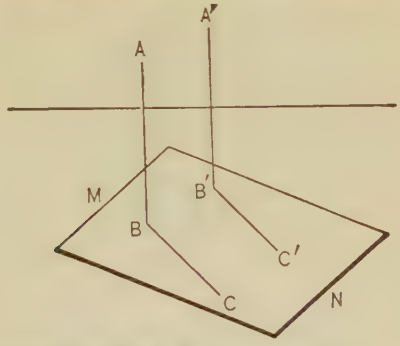
508. Cor. 1. *Every point that is equidistant from the ends of a line lies in the plane that bisects the given line at right angles.*

Hint. Using fig. above, gotten by passing a plane through point $P \perp$ to line AB, we prove that plane RS will bisect AB, and consequently plane RS is the perpendicular bisector of line AB.

509. Cor. 2. *The locus of all points equidistant from the ends of a line is the plane that is perpendicular to the line at its middle point.*

PROPOSITION XI. THEOREM

510. *If one of two parallel lines is perpendicular to a plane, the other is also perpendicular to the plane.*



Given $AB \parallel A'B'$, and $AB \perp$ plane MN.

To prove $A'B' \perp$ plane MN.

Proof. In the plane MN, through B and B' respectively draw any two parallel lines BC and $B'C'$.

Then $\angle ABC = \angle A'B'C'$. (Why?)

But $\angle ABC$ is a rt. $\angle$.

Hence $\angle A'B'C'$ is a rt. $\angle$,

and $A'B'$ is perpendicular to any line drawn through its foot.

Whence $A'B' \perp$ plane MN.

Ex. **1**. The lines joining, in succession, the mid-points of the sides of a quadrilateral in space form a parallelogram.

Ex. **2**. The lines joining the mid-points of opposite sides of any quadrilateral in space bisect each other.

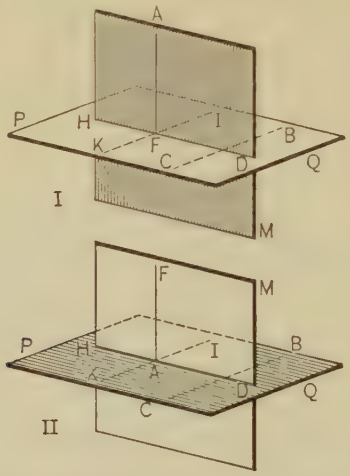
Ex. **3**. State Prop. XI, in the form: If a plane is perpendicular to one of two parallel lines, . . .

Ex. **4**. Prove Prop. XI by drawing BB' and using, in addition, one pair of lines as BC and $B'C'$.

Ex. **5**. Show that if a plane cuts one of two parallel lines obliquely, it is not perpendicular to the other line.

Proposition XII. Problem

511. *Through a given point, to draw a line perpendicular to a given plane.*



Given plane PQ and point A.

Required. Through A, to draw a line perpendicular to PQ.

Construction. In plane PQ, draw any line BC.

Through A construct a plane $AM \perp BC$ intersecting PQ in DH.

Through A in the plane AM draw $AF \perp DH$.

AF is the required perpendicular.

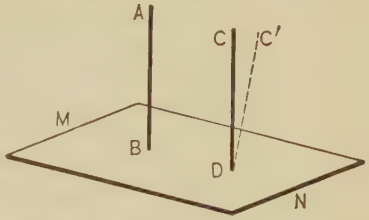
Proof. Through the foot of AF (F in Fig. I and A in Fig. II) in plane PQ draw $IK \parallel BC$.

Since	$BC \perp$ plane AM,	(Con.)
	$BC \parallel IK$.	(Con.)
	$\therefore IK \perp$ plane AM.	(510)
	$\therefore FA \perp IK$.	(Why?)
But	$FA \perp DH$.	(Con.)
	$\therefore FA \perp$ plane PQ.	(Why?)

512. Cor. 1. *Through a given point, one and only one perpendicular can be drawn to a given plane.*

For if there were two perpendiculars passing through the point, their plane would intersect the given plane in a straight line, perpendicular to both perpendiculars. Or, in a plane, there would be two perpendiculars to a given line, through a given point, which is impossible.

513. Cor. 2. *If two lines are perpendicular to the same plane, these lines are parallel.*



Let AB and CD be $\perp$ plane MN.

Draw $DC' \parallel AB$.

Then $DC' \perp MN$.

But through D only one perpendicular can be drawn to plane MN.

Hence DC and DC' coincide, and $DC \parallel AB$.

514. Def. The **distance of a point from a plane** is the length of the perpendicular from the point to that plane.

Ex. **1**. What is the locus of a point having a given distance from a given plane?

Ex. **2**. What is the locus of a point having a given distance from a given plane and equidistant from two given points?

Ex. **3**. Find the locus of points equidistant from points A and B, and also equidistant from points C and D.

Ex. **4**. Discuss Ex. 3 when (*a*) line $AB \parallel$ line CD, (*b*) C coincides with B, (*d*) all the points A, B, C, and D are in the same line.

Ex. **5**. Through a point outside two intersecting planes, draw a line that will be parallel to each plane.

Ex. **6**. Prove: Every triangle determines a plane.

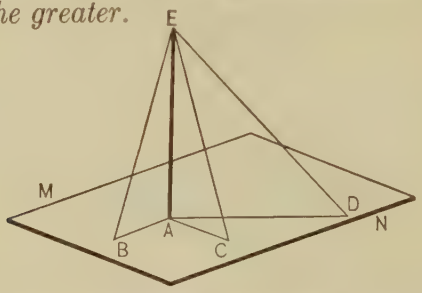
Ex. **7**. If $AB \parallel A'B'$ and through each line a plane is passed intersecting each other in line CD, then $CD \parallel AB$.

Proposition XIII. Theorem

515. *If, from a point without a plane, oblique lines be drawn to the plane:*

(1) *Any two lines meeting the plane at equal distances from the foot of the perpendicular are equal.*

(2) *Of two lines meeting the plane at unequal distances from the foot of the perpendicular, the more remote is the greater.*



Given EA perpendicular to plane MN. Oblique lines EB, EC, and ED are drawn so that $AB = AC$ and $AD > AC$.

To prove (1) $EB = EC$.

(2) $ED > EC$.

Hint. 1. Prove by means of congruent triangles.

2. $\overline{ED}^2 = \overline{EA}^2 + \overline{AD}^2$, $\overline{EC}^2 = \overline{EA}^2 + \overline{AC}^2$.

516. Cor. 1. Conversely, *equal oblique lines drawn from a point to a plane meet the plane at equal distances from the foot of the perpendicular from the point to the plane, and of two unequal oblique lines, the greater one meets the plane at a greater distance from the foot.*

517. Cor. 2. *The locus of a point in space that is equidistant from all the points in a circle is a straight line perpendicular to the plane of the circle, and passing through the center of the circle.*

518. Cor. 3. *The perpendicular to a plane is the shortest line that can be drawn to the plane from a point without and conversely.*

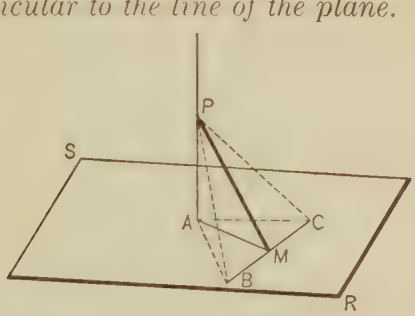
519. Cor. 4. *If two planes are parallel, they are everywhere equally distant and conversely.*

Ex. 1. In the diagram of Prop. XIII, if $\angle B = \angle C$, then $EB = EC$, and $AB = AC$.

Ex. 2. From a point P, 3 in. from a plane MN, a line PA is drawn, A being in MN. If $PA = 5$ in., what is the distance of A from the foot of the perpendicular drawn from P to MN?

Proposition XIV. Theorem

520. *If from the foot of a perpendicular to a plane a straight line is drawn at right angles to any line in the plane, the line drawn from its intersection with the line in the plane to any point of the perpendicular is perpendicular to the line of the plane.*



Given $PA \perp$ pl. SR, $AM \perp BC$ in plane SR, and PM.

To prove $PM \perp BC$.

Proof. Lay off $BM = MC$; draw AB, AC, PB, and PC.

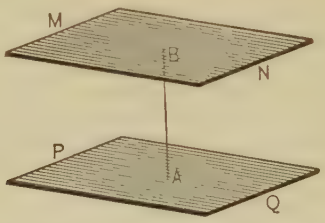
$AB = AC$. (Why?)

$PB = PC$. (515)

$\therefore PM \perp BC$. (Why?)

Proposition XV. Theorem

521. *Planes perpendicular to the same straight line are parallel to each other.*



Given planes MN and PQ perpendicular to line AB.

To prove plane $MN \parallel$ plane PQ.

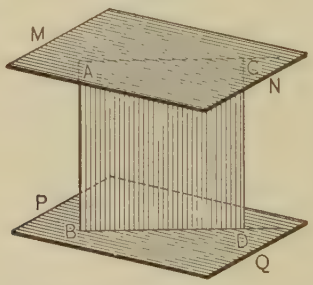
Proof. If planes MN and PI are not parallel, they will meet in some point.

This is impossible (506)

$\therefore$ pl. $MN \parallel$ pl. PQ.

Proposition XVI. Theorem

522. *A straight line perpendicular to one of two parallel planes is perpendicular to the other also.*



Given plane $MN \parallel$ plane PQ, and $AB \perp$ plane MN.

To prove $AB \perp$ plane PQ.

Proof. Through AB pass any plane intersecting MN in AC and PQ in BD, respectively.

$AC \parallel BD$, (486)

and $AC \perp AB$. (500)

Hence $BD \perp AB$. (105)

Therefore AB is perpendicular to any line in PQ passing through B.

Whence $AB \perp$ plane PQ.

523. Cor. *Through a given point, one plane and only one can be passed parallel to a given plane.*

Ex. *If two planes are parallel to a third plane, they are parallel to each other.*

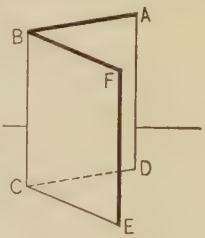
Hint. Construct a line $\perp$ one of the planes.

DIHEDRAL ANGLES

524. If a plane ABC rotates about the line BC until it reaches the position FBC, then the amount of rotation is called a **dihedral angle**.

The line BC is called the **edge** and the planes ABC and FBC are called the **faces** of the dihedral angle.

Note. The student should note that the above statement is not a definition, but merely an explanation, of *dihedral angle.*



525. A dihedral angle may be designated by two letters on its edge; or if several dihedral angles have a common edge, by four letters, one on each face, and two on the edge, the letters on the edge being placed between the others.

Thus, the dihedral angle in the annexed diagram may be designated by BC or A–BC–E.

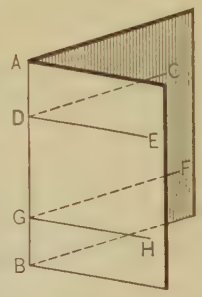
Obviously the size of the dihedral angle does not depend upon the extent of its faces.

526. DEF. A **plane angle of a dihedral angle** is an angle formed by perpendiculars to the edge at the same point, one in each face.

Thus, $\angle CDE$ is a plane angle of dihedral angle AB if CD is perpendicular to AB and ED is perpendicular to AB.

Two plane angles of the same dihedral angle, as CDE and FGH, are equal, since their sides are respectively parallel and lie in the same direction.

Or, the plane angle of a dihedral angle is the same, at whatever point of the edge it is drawn.



527. Two **dihedral angles are equal** when they can be made to coincide.

By superposing two equal dihedral angles, their plane angle can be made to coincide. Hence equal dihedral angles have equal plane angles.

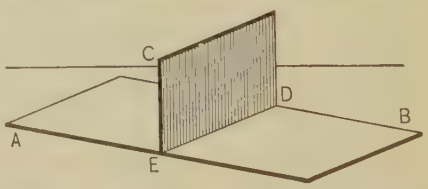
528. A dihedral angle is **acute**, **right**, **obtuse**, or **straight**, according as its plane angle is acute, right, obtuse, or straight.

It is evident that the two faces of a straight dihedral angle lie in the same plane.

Dihedral angles are **complementary**, **supplementary**, **adjacent**, **vertical**, etc., according as their plane angles are complementary, supplementary, adjacent, etc.

If the plane meets another plane so as to make the two adjacent dihedrals equal, each dihedral is evidently a right one.

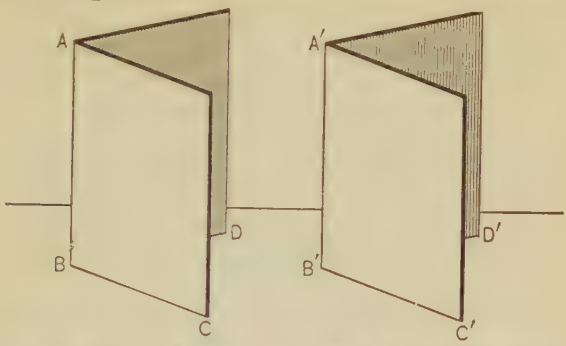
529. A straight dihedral angle is divided into 180 equal parts, each called a **degree**. Degrees are divided into **minutes**, minutes into **seconds**.



530. Planes forming right dihedral angles are said to be **perpendicular to each other**. Thus, plane CD is perpendicular to plane AB if they form right dihedral angles.

PROPOSITION XVII. THEOREM

531. *Two dihedral angles are equal if their plane angles are equal.*



Given the dihedral angles AB and $A'B'$ with equal plane angles CBD and $C'B'D'$ respectively.

To prove dihedral angle AB = dihedral angle $A'B'$.

Proof. Since $AB \perp BD$, and $AB \perp BC$,

$AB \perp$ plane BDC.

And, similarly, $A'B' \perp$ plane $B'D'C'$.

Apply $\angle DBC$ to $\angle D'B'C'$.

Then plane BDC coincides with plane $B'D'C'$.

Hence BA coincides with $B'A'$, (512)

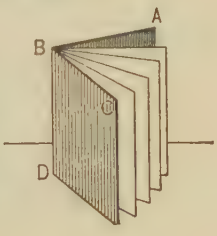
and since also DB coincides with $D'B'$,

plane AD coincides with plane $A'D'$. (484)

In like manner plane AC coincides with plane $A'C'$.

Hence, dihedral angles AB and $A'B'$ coincide.

Therefore they are equal.

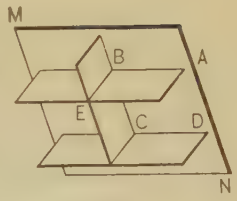


532. COR. If a dihedral angle is divided into n equal parts, its plane angle is divided into n equal parts; and conversely.

Ex. 1. *Vertical dihedral angles are equal.*

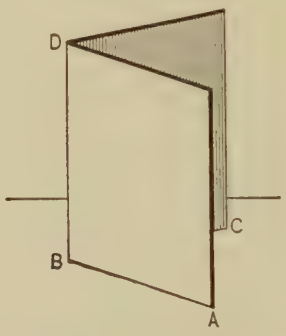
Ex. 2. If two parallel planes are intersected by a third one, the corresponding dihedral angles are equal.

Hint. Construct a plane $MN \perp EB$, the intersection of two planes.



Proposition XVIII. Theorem

533. *A dihedral angle is measured by its plane angle.*



Given $\angle ABC$, the plane angle of dihedral $\angle BD$.

To prove $\angle ABC$ and dihedral $\angle BD$ are measured by the same number of degrees.

Proof. Case I. The number of degrees contained in $\angle ABC$ is a rational number, *e.g.* $\frac{m}{n}$, where m and n are integers.

A dihedral angle of 180° has a plane angle of 180°. (527)

$\therefore$ a dihedral angle of 1° has a plane angle of 1°. (532)

$\therefore$ a dihedral angle of $\left(\frac{1}{n}\right)°$ has a plane angle of $\left(\frac{1}{n}\right)°$. (532)

$\therefore$ a dihedral angle of $\left(\frac{m}{n}\right)°$ has a plane angle of $\left(\frac{m}{n}\right)°$.

CASE II. The number of degrees in $\angle ABC$ is an irrational number.

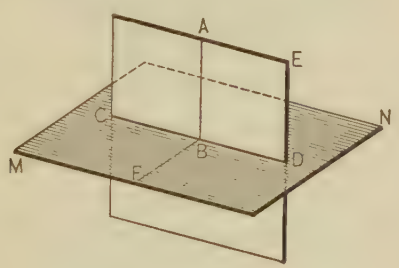
Any approximation of $\angle ABC$, being a rational number, must equal the corresponding approximation of BD (Case I).

Or, *all* approximations of the numerical measures of ABC and BD are respectively equal. Hence these numerical measures are equal. (223)

EX. If two adjacent dihedral angles are respectively 30° and 40°, find the dihedral angle formed by the planes which bisect these dihedral angles.

PROPOSITION XIX. THEOREM

534. *If a straight line is perpendicular to a plane, every plane passed through this line is perpendicular to the plane.*



Given $AB \perp$ plane MN, and plane EC containing AB.

To prove plane $EC \perp$ plane MN.

HINT. In plane MN at B, draw $BF \perp CD$, and prove that $\angle ABF$ is a right angle.

535. NOTE. The preceding proposition furnishes the usual method for constructing a plane perpendicular to a given plane. In such constructions draw first a *line* perpendicular to the given plane, and then pass a plane through it.

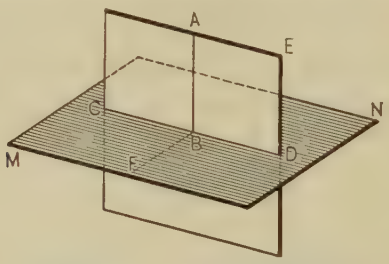
Ex. 1. Through a given point to construct a plane perpendicular to two given planes.

Ex. 2. Through a given point to construct a plane perpendicular to a given plane P and parallel to a given line CD.

Ex. 3. Through a given straight line to construct a plane perpendicular to a given plane.

Proposition XX. Theorem

536. *If two planes are perpendicular to each other, a straight line drawn in one of them, perpendicular to their intersection, is perpendicular to the other.*



Given plane $CE \perp$ plane MN, and in CE line $BA \perp CD$, the intersection of the two planes.

To prove $AB \perp$ plane MN.

Hint. At B construct the plane angle and prove that $AB \perp$ two lines drawn through its foot.

537. Cor. 1. *If two planes are perpendicular to each other, a perpendicular to one of these, at any point of their intersection, lies in the other.*

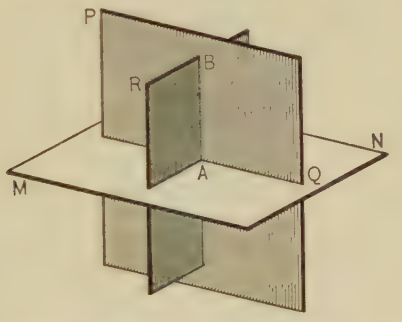
Let BP be $\perp$ pl. MN at point B. Draw BA as in fig. above. Then BP coincides with BA. Hence BP lies in plane CE.

538. Cor. 2. *If two planes are perpendicular to each other, a straight line drawn from any point of one, perpendicular to the other, lies in the first.* [Prove by indirect method.]

Ex. In the diagram for Prop. XX, find the length of AF, if plane $CE \perp$ plane MN, $AB \perp CD$, $AB = m$ in., and $BF = n$ in.

Proposition XXI. Theorem

539. *If two intersecting planes are each perpendicular to a third plane, their intersection is perpendicular to that plane.*



Given the planes PQ and $RA \perp$ plane MN, and intersecting in AB.

To prove $AB \perp$ plane MN.

Proof. At A draw a straight line $\perp MN$.

This line must lie in plane PQ and in plane RA. (537)

Hence it must coincide with AB.

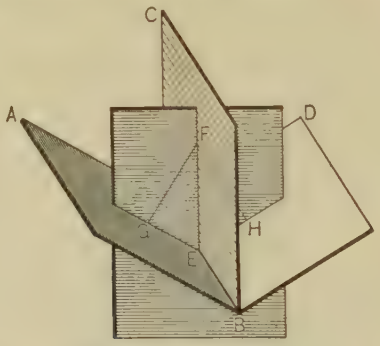
Or $AB \perp$ plane MN.

540. Cor. *If two planes are perpendicular to each other, and a third plane is perpendicular to their intersection, each of the three lines of intersection is perpendicular to the other two.*

Ex. If a line is parallel to a plane, any plane perpendicular to the line is perpendicular to the first plane.

Proposition XXII. Theorem

541. *Every point in the plane bisecting a dihedral angle is equidistant from the faces of the angle.*



Given plane CB bisecting dihedral $\angle A\text{–}BE\text{–}D$, and FG and FH the distances of a point F in BC, from AB and BD respectively.

To prove $FH = FG$.

Proof. Through FG and FH pass a plane intersecting the faces in EG and EH.

Plane $FGH \perp AB$ and DB. (534)

Therefore plane $FGH \perp BE$. (539)

Hence $EB \perp EG$, EF, and EH. (500)

$\therefore \angle GEF$ and $\angle HEF$ are plane angles of dihedral angles $A\text{–}BE\text{–}C$ and $D\text{–}BE\text{–}C$.

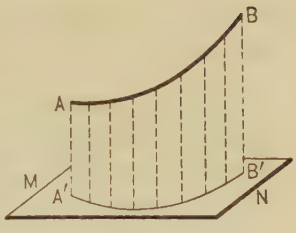
[To be completed by the student. Compare 70.]

542. Cor. *Every point equidistant from the faces of a dihedral angle lies in the plane bisecting the dihedral angle.*

Ex. The locus of a point equidistant from two intersecting planes consists of the two planes bisecting the dihedral angles formed by the planes.

543. DEF. The **projection of a point** on a plane is the foot of the perpendicular from the point to the plane.

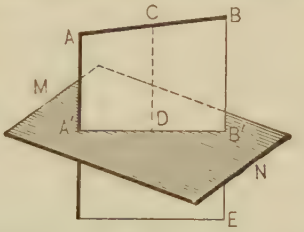
The **projection of a figure** upon a plane is the locus of the projections of all the points of the figure upon the plane.



Thus, $A'B'$ represents the projection of AB upon plane MN.

PROPOSITION XXIII. THEOREM

544. *Through any given straight line not perpendicular to a plane, one plane can be passed perpendicular to the given plane, and only one.*



Given AB a line not $\perp$ plane MN.

To prove through AB one, and only one, plane can be passed $\perp$ MN.

Proof. From any point C in AB, draw $CD \perp MN$.

Pass a plane AE through AB and CD.

Then plane AD is $\perp$ MN. (534)

If two planes could be passed through $AB \perp$ plane MN, AB, their intersection, would be perpendicular to plane MN. (539)

But this is impossible since it contradicts the hypothesis. Hence, only one plane can be drawn perpendicular to plane MN and passing through AB.

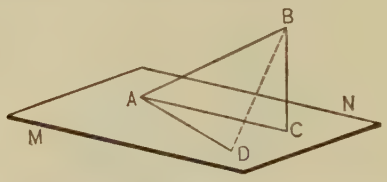
545. Cor. *The projection of a line not perpendicular to a plane upon that plane is a straight line.*

546. Def. The **inclination of a line to a plane**, or the **angle made by a line and a plane**, is the angle which the line forms **with its own projection upon that plane.**

Thus, in the diagram for the next proposition, if AC is the projection of AB upon MN, then $\angle BAC$ is the angle formed by AB and MN.

Proposition XXIV. Theorem

547. *The acute angle formed by a line and its projection upon a plane is the least angle that the line makes with any line in the plane.*



Given the line AC, the projection of line AB upon plane MN, and AD, any other line drawn through A in MN.

To prove $\angle BAC < \angle BAD.$

Proof. In plane MN make $AD = AC$ and draw BD.

In $\triangle ABC$ and ABD,

$$AB = AB,$$

$$AC = AD,$$

$$BC < BD. \qquad (518)$$

Hence $\angle BAC < \angle BAD.$ (133)

Ex. **1**. Two planes are parallel, and a circle in one is projected upon the other. What is the projection? What is the projection if the planes are perpendicular to each other?

Ex. **2**. If the projection of a figure upon a plane is a straight line, the figure is a plane figure.

Ex. **3**. If a line AB and a plane MN form an angle of 45°, and $AB = 10$ in., what is the length of the projection of AB upon MN?

Ex. **4**. In the diagram for Prop. XXIII find the inclination of AB to MN, if $AA' = 6$, $BB' = 11$, and $AB = 10$.

Ex. **5**. In the diagram for Prop. XXIV, if $AC = 3$ in., $\angle BAC = 60°$, and $\angle CAD = 90°$, $AC = AD$, and $\angle C = 90°$, find the length of BD.

Ex. **6**. *The inclinations of two parallel lines to a plane are equal.*

Ex. **7**. In the diagram for Prop. XXIV (but having $AC \neq AD$), find $\angle BAD$, if $AB = 10$, $CD = 5\sqrt{2}$, $\angle BAC = 30°$, and $\angle CDA = 90°$.

Ex. **8**. In a similar diagram find AB and BD, if $AD = m$, $DC = n$, $CB = p$, and $\angle ADC = 90°$. Derive from these the value of $\angle ADB$.

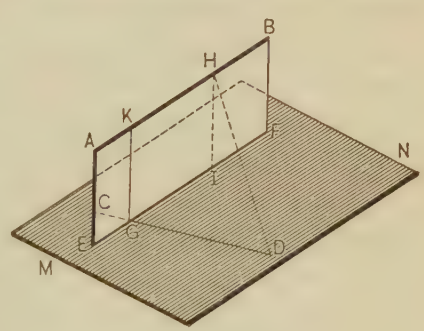
Ex. **9**. *To draw a common perpendicular to two lines AB and CD, not in the same plane.*

HINT. Through CD draw plane $MN \parallel AB$.

Through AB draw plane $AF \perp MN$.

Draw $GK \perp EF$.

GK is the required line.



Exs. 9, 10

Ex. **10**. Prove only one such common perpendicular can be drawn. [Use fig. above.]

Ex. **11**. Find the locus of a point equidistant from three given points not in a straight line.

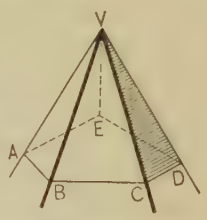
Ex. **12**. If a line meets its projection on a plane, any line in the plane perpendicular to one of them at their intersection is perpendicular to the other also.

POLYHEDRAL ANGLES

548. DEF. A **polyhedral angle** is the figure generated by a moving straight line which continually intersects a fixed polygon, and terminates in a fixed point not in the plane of the polygon.

Thus, *V–ABCDE* is a polyhedral angle.

549. The **vertex** of the polyhedral angle is the fixed point, as *V*. An **element** is the moving line in any position. An element through one of the vertices of the polygon is an **edge**; as *VA*, *VB*, *VC*, etc. The parts of a plane determined by two successive edges is a **face**; as *AVB*, *BVC*, etc. A **face angle** of the polyhedral angle is the angle formed by two successive edges; as $\angle AVB$, $\angle BVC$, etc. A **dihedral angle** of the polyhedral angle is the dihedral angle formed by two adjacent faces; as dihedral $\angle VA$, $\angle VB$, etc.



550. A polyhedral angle is **convex**, if the given polygon is convex.

551. DEF. A polyhedral angle is called **trihedral**, **tetrahedral**, etc., according as it has three, four, etc., faces.

552. DEF. A trihedral angle is called **rectangular**, **birectangular**, or **trirectangular**, according as it has one, two, or three right dihedral angles.

553. DEF. An **isosceles** trihedral angle is one having two of its face angles equal.

554. Two *polyhedral angles are congruent* if the face angles and the dihedral angles of the one are respectively equal to the face angles and the dihedral angles of the other one, and all the parts are arranged in the same order (for evidently they can be made to coincide).

Two *polyhedral angles are symmetric* if the face angles and the dihedral angles of the one are respectively equal to the face angles and the dihedral angles of the other, and all parts are arranged in reverse order.

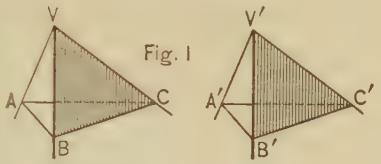


Fig. 1

Thus, in Fig. 1 the trihedral $\angle V\text{-}ABC \cong$ trihedral $\angle V'\text{-}A'B'C'$ if $\angle AVB = \angle A'V'B'$, $\angle BVC = \angle B'V'C'$, $\angle CVA = \angle C'V'A'$, and dihedral $\angle AV$ = dihedral $\angle A'V'$, dihedral $\angle BV$ = dihedral $\angle B'V'$, and dihedral $\angle CV$ = dihedral $\angle C'V'$.

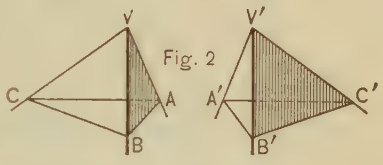


Fig. 2

While in Fig. 2, $V\text{-}ABC$ and $V'\text{-}A'B'C'$ are symmetric.

It is obvious that, in general, two symmetric polyhedral angles cannot be made to coincide.

Proposition XXV. Theorem

555. *Two trihedral angles are congruent if two face angles and the included dihedral angle of the one are respectively equal to two face angles and the included dihedral angle of the other, the equal parts being arranged in the same order.*

[Proof by superposition.]

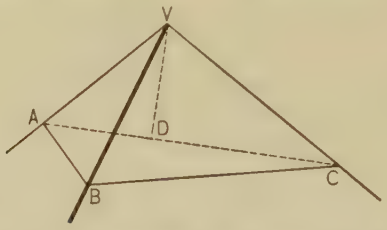
Proposition XXVI. Theorem

556. *Two trihedral angles are congruent if two dihedral angles and the included face angle of the one are respectively equal to two dihedral angles and the included face angle of the other, the equal parts being arranged in the same order.*

[Proof by superposition.]

Proposition XXVII. Theorem

557. *The sum of any two face angles of a trihedral angle is greater than the third face angle.*



Given $\angle AVC$, the greatest face angle of trihedral angle $V\text{–}ABC$.

To prove $\angle AVB + \angle BVC > \angle AVC$.

Proof. In the face angle AVC draw AC, then draw VD making $\angle DVA = \angle BVA$.

Lay off $VB = VD$. Draw BC.

Then $\triangle AVB \cong \triangle AVD$. (Why?)

$\therefore AB = AD$.

But $AB + BC > AC$. (Why?)

$\therefore BC > DC$. (Ax. 5.)

But in the $\triangle BVC$ and DVC,

VC is common,

$VD = VB$, (Why?)

and $BC > DC$.

Hence $\angle BVC > \angle DVC$. (133)

Adding the equal angles AVB and AVD,

$$\angle AVB + \angle BVC > \angle DVC + \angle AVD,$$

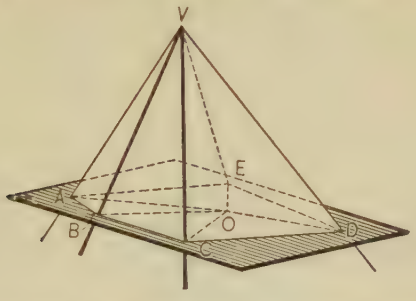
or $$\angle AVB + \angle BVC > \angle AVC.$$

Ex. 1. Any one of the plane angles forming a polyhedral angle is less than the sum of the other plane angles.

Ex. 2. The sum of the angles of a skew (non-planar) quadrilateral is less than four right angles.

PROPOSITION XXVIII. THEOREM

558. *The sum of the face angles of any convex polyhedral angle is less than four right angles.*



Given V–$ABCDE$, any convex polyhedral angle.

To prove the sum of $\measuredangle AVB$, BVC, etc., is less than four right angles.

Proof. Construct a plane intersecting the edges in A, B, C, etc., the faces in AB, BC, CD, etc. Then $ABCDE$ is a convex polygon.

Join any point O in plane ABC to A, B, C, etc.

Then $\angle VBA + \angle VBC > \angle ABC$,

and $\angle VCB + \angle VCD > \angle BCD$, etc. (557)

Therefore the sum of all the base angles of the triangles whose vertex is V is greater than the sum of all the base angles of the triangles whose vertex is O.

But the sum of all the angles of the triangles whose vertex is V is equal to the sum of all the angles of the triangles whose vertex is O. (Why?)

Subtracting the base angles from the equal sums, the sum of all the angles at V is less than the sum of all the angles at O.

But the sum of all the angles at O is equal to four right angles.

Whence, the sum of all the angles at V is less than four right angles.

Proposition XXIX. Theorem

559. *If two trihedral angles have the three face angles of one equal respectively to the three face angles of the other, the corresponding dihedral angles are equal and the trihedral angles are either equal or symmetrical.*

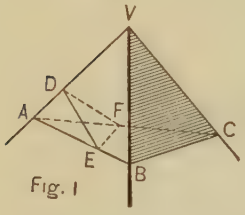


Fig. 1

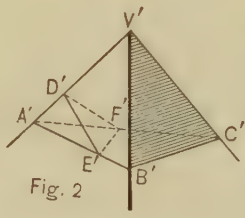


Fig. 2

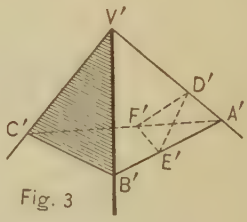


Fig. 3

Given in trihedral angles V–ABC and V'–$A'B'C'$,
$\angle AVB = \angle A'V'B'$, $\angle BVC = \angle B'V'C'$, $\angle CVA = \angle C'V'A'$.

To prove V–$ABC \cong$ or sym. to V'–$A'B'C'$ and dih. $\measuredangle$ VA, VB, and VC equal respectively to $V'A'$, $V'B'$, $V'C'$.

Proof. On the six edges lay off

$$VA = VB = VC = V'A' = V'B' = V'C'.$$

Draw AB, BC, CA, $A'B'$, $B'C'$, and $C'A'$.

Then $\triangle AVB \cong \triangle A'V'B'$, (Why?)

and $AB = A'B'$.

In like manner, $BC = B'C'$, $CA = C'A'$.

On AV and $A'V'$, respectively, lay off AD equal to $A'D'$. Draw DE in face AVB, and DF in face AVC, perpendicular to VA. These lines meet AB and AC, respectively, since triangles AVB and AVC are isosceles.

In like manner, draw $D'E'$ and $D'F'$, and join EF and $E'F'$. [To be completed by the student.]

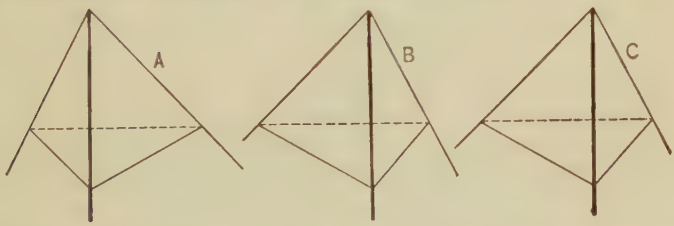
HINT. Prove $\triangle ABC \cong \triangle A'B'C'$.
$\triangle ADE \cong \triangle A'D'E'$.
$\triangle ADF \cong \triangle A'D'F'$.
$\triangle AEF \cong \triangle A'E'F'$.
$\triangle DEF \cong \triangle D'E'F'$.

560. DEF. **Two polyhedral angles** are **vertical** if the edges of each are prolongations of the edges of the other through the vertex.

PROPOSITION XXX

561. *Two trihedral angles are symmetric:*

(1) *If two face angles and the included dihedral angle of the one are respectively equal to two face angles and the included dihedral angle of the other,*
or, (2) *If two dihedral angles and the included face angle of the one are respectively equal to two dihedral angles and the included face angle of the other,*
or, (3) *Three face angles of the one are equal respectively to three face angles of the other,*
provided all equal parts are arranged in reverse order.



Proof. If A and B are the given trihedral angles, construct trihedral angle C symmetric to A.

Since C and A have all parts arranged in reverse order, C and B have all parts arranged in the same order.

Hence $C \cong B$.

But since C is symmetric to A, the congruent figure B is symmetric to A.

562. Cor. *Vertical trihedral angles are symmetric.*

---

563. Note. There is a remarkable analogy between triangles and trihedral angles (also between polygons and polyhedral angles). The sides of a triangle correspond to the face angles, and the angles of a triangle correspond to the dihedral angles of a trihedral angle. Proofs of theorems that relate to trihedral angles may often be found from the proofs of the analogous triangle proposition by the addition of the letter V (*i.e.* the vertex) to each symbol. Thus, the following statement, if we erase everywhere the letter V, is the proof of the proposition:

If two sides (AB and BC) of a triangle (ABC) are equal, the opposite angles (A and B) are equal. If we include the letter V, it is the proof for the theorem:

If two face angles of a trihedral angle are equal, the opposite dihedral angles are equal.

Ex. **Given** $\angle AVB = \angle BVC$.

To prove dih. $\angle VA$ = dih. $\angle VC$.

Proof. Draw plane VBD bisecting $\angle VB$.

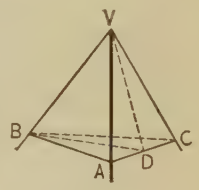
$\angle AVB = \angle BVC$. (Hyp.)

$A\text{-}VB\text{-}D = C\text{-}VB\text{-}D$. (Con.)

$\angle BVD = \angle BVD$. (Iden.)

$\therefore$ $V\text{-}ABD$ is symmetric to $V\text{-}BDC$.

$\therefore$ dih. $\angle VA$ = dih. $\angle VC$.



EXERCISES

Ex. **1**. *State* propositions relating to trihedral or polyhedral angles that correspond to the following theorems of plane geometry:

(*a*) The sum of two sides of a triangle is greater than the third side.

(*b*) If two angles of a triangle are equal, the opposite sides are equal.

(*c*) If two sides of a triangle are unequal, the opposite angles are unequal, etc.

(*d*) Two triangles are congruent if two sides and the included angle of the one, etc.

(*e*) If the opposite sides of a quadrilateral are equal, the opposite angles are equal.

Ex. **2.** In the diagram for Prop. XXVII,

$$\angle AVD + \angle DVB < \angle AVC + \angle CVB.$$

Ex. **3.** If in tetrahedral angle $V\text{–}ABCD$, $\angle AVB = \angle AVD$ and $\angle BVC = \angle CVD$, then dihedral angle VD = dihedral angle VB.

Ex. **4.** If the opposite face angles of a tetrahedral angle are equal, the opposite dihedral angles are equal.

Ex. **5.** *The projections of parallel lines upon the same plane are parallel.*

Ex. **6.** Planes bisecting supplementary adjacent dihedral angles are perpendicular to each other.

Ex. **7.** A line is twice as long as its projection upon a plane. What is the inclination of the line to the plane?

Ex. **8.** If the three face angles of a trihedral angle are equal, the three dihedral angles are equal.

Ex. **9.** If the three face angles of a trihedral angle are right angles, the three dihedral angles are right ones (a trirectangular trihedral angle).

Ex. **10.** *If a straight line intersects two parallel planes, it makes equal angles with both planes.*

Ex. **11.** *Vertical polyhedral angles are symmetric.*

Ex. **12.** Find the locus of a point equidistant from the three faces of a trihedral angle.

Ex. **13.** *Find a point equidistant from four given points, not in one plane.*

Ex. **14.** What is the locus of a point X, if X lies in plane MN, PX equals a given line, and P is a point without MN?

Ex. **15.** If the sum of the four angles of a quadrilateral is equal to four right angles, its vertices lie in a plane.

Ex. **16.** If D is a point within a trihedral angle $V\text{–}ABC$, then $\angle AVD + \angle BVD + \angle CVD > \frac{1}{2}(\angle AVB + \angle BVC + \angle CVA)$.

Ex. **17.** Through a given point draw a line parallel to a given plane.

Ex. **18.** When will a right angle project into a right angle? An acute angle? A straight angle?

Ex. **19.** In the example under section 563, prove that plane BVD is perpendicular to plane VAC.

REVIEW EXERCISES

Ex. 1. Two lines are parallel to the same plane. Are they parallel to each other? Why?

Ex. 2. If a line and a plane are perpendicular to the same plane, they are parallel, or the line lies in the first plane.

Ex. 3. If a plane is passed through one of the diagonals of a parallelogram, the perpendiculars to the plane from the extremities of the other diagonal are equal.

Ex. 4. Given a point and two nonintersecting straight lines in space, show that in general one and only one straight line can be drawn through the point to meet the lines. Mention exceptional cases.

Ex. 5. Prove that if a line is perpendicular to one of two intersecting planes, its projection on the other is perpendicular to the line of intersection of the two planes.

Ex. 6. Find the locus of all points that are within a given trihedral angle and equally distant from the edges.

Ex. 7. Prove that if from any point in a dihedral angle perpendiculars are drawn to the faces, the plane determined by these perpendiculars is perpendicular to the edge of the dihedral angle.

Ex. 8. Three plane angles are 20°, 80°, and 105°. Will these angles form a trihedral? Why?

Ex. 9. What is the locus of all points equidistant from two intersecting lines? From two parallel lines? From two lines, neither parallel nor intersecting?

Ex. 10. From any point within a dihedral angle perpendiculars are drawn to each of the faces. Prove that the angle formed by the two lines is supplementary to the plane angle of the dihedral.

Ex. 11. Prove that if two parallel line-segments are projected on the same plane, their projections have the same ratio as the line-segments themselves.

Ex. 12. Parallel planes cutting off equal distances on one transversal cut off equal distances on every transversal.

Ex. 13. To pass a plane cutting the faces of an acute dihedral angle, so that the intersections form a right angle.

Ex. 14. To cut the four faces of a polyhedral angle of four faces by a plane, so that the section is a parallelogram.

Ex. **15**. To find a point in a plane that shall be equidistant from three given points in space.

Ex. **16**. In a given plane, to determine a point such that the sum of its distances from two points on the same side of the plane shall be a minimum.

Ex. **17**. What is the locus of a point such that the sum of its distances from two given planes is equal to a given line-segment?

Ex. **18**. Given three lines in space. Draw a straight line from the first to the second and parallel to the third.

Ex. **19**. Prove: If a straight line is parallel to a plane, it is everywhere equidistant from the plane.

Ex. **20**. If a line is drawn in the plane of each face of any trihedral angle through its vertex and perpendicular to the opposite edge, prove these three lines lie in the same plane.

Ex. **21**. On one wall of a rectangular room is a point A and on the opposite wall is a point B. State without proof, what points on the floor are equidistant from A and B.

Ex. **22**. Can a straight line be perpendicular to each of two intersecting planes? Why?

Ex. **23**. If a straight ruler be held parallel to a blackboard, will its shadow on the blackboard be parallel to the ruler? Why?

Ex. **24**. A and B are points 8 inches apart. What is the locus of all points equidistant from A and B and also equidistant from two parallel planes that are 10 inches apart?

Ex. **25**. What is the locus of a point equidistant from two parallel planes?

Ex. **26**. Through each point in a straight line, lines are drawn parallel to a given line. Show that these lines all lie in the same plane.

Ex. **27**. The bisectors of the dihedral angles of a trihedral angle intersect in a straight line.

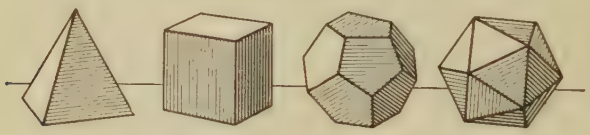
Ex. **28**. A triangle has its base of 6″ lying in a plane. The altitude to this base is 4″. If the triangle makes an angle of 60° with the plane, what is the area of its projection on this plane?

BOOK VII

POLYHEDRONS, CYLINDERS, AND CONES

POLYHEDRONS

564. DEF. A **polyhedron** is the closed figure formed by plane polygons. The **faces** of a polyhedron are the bounding plane polygons; the **edges** are the intersections of the faces; and the **vertices** are the intersections of the edges.



565. DEF. A **tetrahedron** is a polyhedron of four faces; a **hexahedron**, one of six faces; an **octahedron**, one of eight faces; a **dodecahedron**, one of twelve faces; an **icosahedron**, one of twenty faces.

NOTE. The least number of faces that a polyhedron can have is *four;* for *three* planes intersecting in a common point form a *trihedral angle*, and, therefore, one more plane is needed to form a solid.

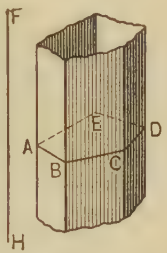
566. DEF. A **diagonal** of a polyhedron is the straight line joining any two vertices not in the same face.

567. DEF. A **convex** polyhedron is one every section of which is a convex polygon.

NOTE. All the polyhedrons treated of in this book are convex.

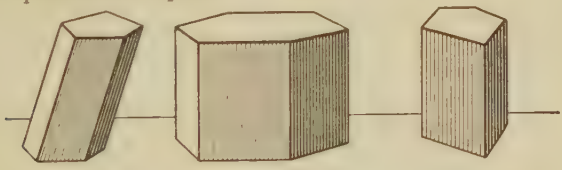
PRISMS AND PARALLELOPIPEDS

568. DEF. A **prismatic surface** is a surface generated by a moving straight line which continually intersects a given polygon (as *ABCDE*) and is parallel to a fixed straight line not in the plane of the polygon (as *FH*).



569. DEF. A **prism** is a polyhedron formed by a prismatic surface and two parallel planes cutting the prismatic surface. The faces made by the parallel planes are the **bases**; the faces made by the prismatic surface are the **lateral faces**. The **lateral edges** are the intersections of two consecutive lateral faces. The **lateral area** is the sum of the areas of the lateral faces. The **altitude** of a prism is the perpendicular distance between the planes of the bases.

Obviously all lateral faces are parallelograms (486) and the lateral edges are parallel and equal.



570. DEF. A **right prism** is a prism whose lateral edges are perpendicular to the planes of the bases.

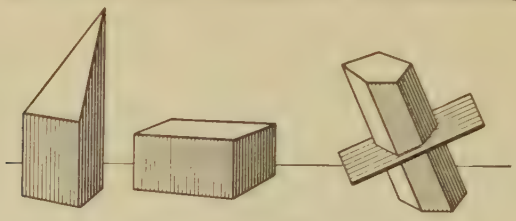
In a right prism all lateral faces are rectangles which are perpendicular to the base, and all lateral edges are equal to the altitude.

571. DEF. A **regular prism** is a right prism whose bases are *regular polygons*.

572. DEF. An **oblique prism** is a prism whose lateral edges are not perpendicular to the planes of the bases.

A prism is **triangular**, **quadrangular**, etc., according as its bases are triangles, quadrilaterals, etc.

573. DEF. A **truncated prism** is the part of a prism included between a base and a section made by a plane not parallel to the base and cutting all of the lateral edges.



574. DEF. A **right section** of a prism is a section made by a plane perpendicular to all the lateral edges of the prism (prolonged if necessary).

If a plane is perpendicular to *one* lateral edge of a prism, it is perpendicular to every lateral edge (510).

575. DEF. A **parallelopiped** is a prism whose bases are parallelograms; hence all of its faces are parallelograms.

576. DEF. A **right parallelopiped** is a parallelopiped whose lateral edges are perpendicular to the bases.

Obviously a parallelopiped is a right parallelopiped if *one* of its lateral edges is perpendicular to the base.

577. DEF. A **rectangular parallelopiped** (rectangular solid) is a right parallelopiped whose bases are rectangles; hence all of its faces are rectangles.

To prove that a parallelopiped is rectangular, it is only necessary to show that the three angles at any one vertex are right angles.

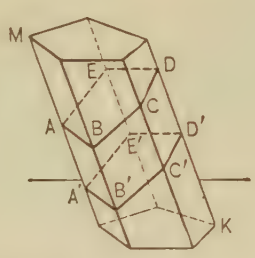
578. DEF. A **cube** is a parallelopiped all of whose faces are squares.

Ex. 1. How many diagonals can be drawn in (*a*) a parallelopiped? (*b*) A pentagonal prism? (*c*) A quadrangular pyramid?

Ex. **2.** What is the least number of faces that a polyhedron can have? The least number of edges? Of vertices?

Proposition I. Theorem

579. *The sections of a prism made by parallel planes cutting all the lateral edges are congruent polygons.*



Given KM, a prism intersected by parallel planes AD and $A'D'$, cutting all the lateral edges.

To prove section $AD \cong$ section $A'D'$.

Proof.

$AB \parallel A'B'$, $BC \parallel B'C'$, etc. (486)

$\angle ABC = \angle A'B'C'$, $\angle BCD = \angle B'C'D'$, etc. (498)

$AB = A'B'$, $BC = B'C'$, $CD = C'D'$, etc. (141)

$\therefore$ section $AB \cong$ section $A'B'$. (155)

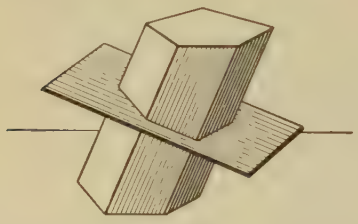
580. Cor. *The bases of a prism are congruent, and every section of a prism made by a plane parallel to the base is congruent to the base.*

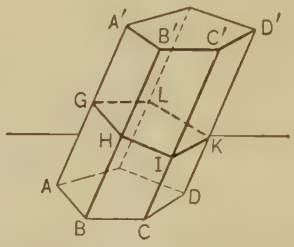
Ex. **1.** Prove, in fig. for Prop. I, that planes through any two non-consecutive lateral edges cut out parallelograms from the prism.

Ex. **2.** Prove, in same figure, that any plane parallel to an edge and cutting the prism, must cut out a parallelogram from the prism.

Proposition II. Theorem

581. *The lateral area of a prism is equal to the product of the perimeter of a right section and a lateral edge.*





Given AD' a prism, L its lateral area, E a lateral edge, and P the perimeter of right section GK.

To prove $L = P \times E$.

Proof. $BB' \perp GH$. (500)

$\therefore$ area $AB' = GH \times BB' = GH \times E$. (353)

Similarly area $BC' = HI \times E$, etc.

But $L =$ area $AB' +$ area $BC' + \cdots$

$= GH \times E + HI \times E + IK \times E + \cdots$

$= (GH + HI + IK + \cdots) \times E$

$= P \times E$.

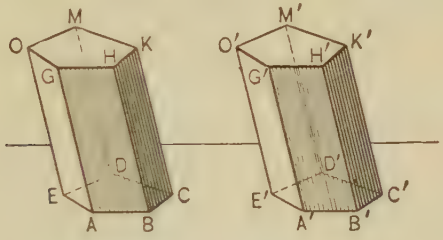
582. Cor. *The lateral area of a right prism equals the perimeter of the base multiplied by the altitude.*

Ex. 1. Find the lateral area of a right prism if its altitude is 15 in., and its base is a triangle the sides of which are 8 in., 10 in., and 11 in. respectively.

Ex. 2. Find the altitude of a right prism if its lateral area equals 190 sq. in., and its base is a quadrilateral the sides of which are 7 in., 8 in., 11 in., and 12 in. respectively.

Proposition III. Theorem

583. *If two prisms have the three faces including a trihedral angle of one respectively congruent to the three faces including a trihedral angle of the other, and the faces similarly placed, they are congruent.*



Given the faces AD, AH, AO, of prism $AM \cong$ the faces $A'D'$, $A'H'$, $A'O'$ respectively of prism $A'M'$, and similarly placed.

To prove $AM \cong A'M'$.

Proof. $\angle BAE = \angle B'A'E'$, $\angle BAG = \angle B'A'G'$, $\angle EAG = \angle E'A'G'$. (Why?)

$\therefore$ trihedral angle $A \cong$ trihedral angle A'. (559)

Place trihedral $\angle A$ on $\angle A'$.

Then face AD coincides with face $A'D'$, face AH with face $A'H'$, and face AO with face $A'O'$.

The point C falls at C', and D at D'. (Why?)

CK will fall on $C'K'$, and DM on $D'M'$. (Why?)

The points G, H, and O coincide respectively with G', H', and O'. (Why?)

$\therefore$ planes GM and $G'M'$ coincide. (481, Post. A.)

Hence point K coincides with K', and M with M'. (Why?)

$\therefore$ the prisms coincide throughout and are congruent.

584. Cor. 1. *Two right prisms having equal bases and equal altitudes are congruent.*

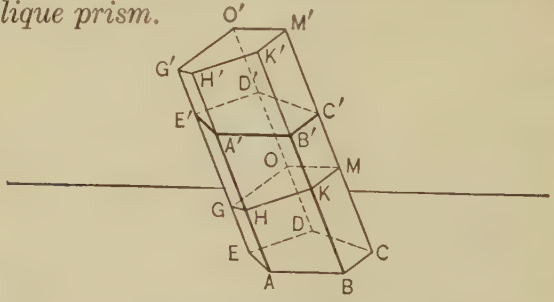
585. Cor. 2. *Two truncated prisms are congruent when they fulfill the hypothesis of the above theorem.*

586. Def. The **volume** of any solid is the number of times it contains the unit of volume. The **unit of volume** is a cube whose edges are equal to the linear unit.

587. Def. **Equivalent** or **equal solids** are solids whose volumes are equal.

Proposition IV. Theorem

588. *An oblique prism is equivalent to a right prism whose base is a right section of the oblique prism, and whose altitude is equal to a lateral edge of the oblique prism.*



Given GM, a right section of oblique prism AD', and GM', a right prism whose altitude is equal to a lateral edge of AD'.

To prove $AD' = GM'$.

Proof. In quadrilaterals $ABKH$ and $A'B'K'H'$,

$AH = A'H', BK = B'K'.$ (Ax. 3)

$AB = A'B', HK = H'K'.$ (139)

$\angle A = \angle A', \angle B = \angle B', \angle K = \angle K', \angle H = \angle H'.$ (106)

$\therefore ABKH \cong A'B'K'H'.$ (155)

Similarly $BKMC \cong B'K'M'C'$.

But face $AD \cong$ face $A'D'$. (580)

$\therefore$ truncated prism $AM \cong$ truncated prism $A'M'$. (585)

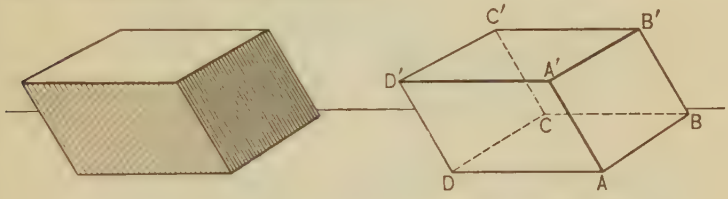
Hence $AM'-AM = AM'-A'M'$, (Ax. 3)

or $AD' = GM'$.

589. Cor. *Prisms having congruent right sections and equal lateral edges are equivalent.*

Proposition V. Theorem

590. *The opposite lateral faces of a parallelopiped are congruent and parallel.*



Given AB' and DC', opposite faces of parallelopiped AC'.

To prove face $AB' \cong DC'$ and face $AB' \parallel DC'$.

Proof.

$AB =$ and $\parallel DC$. (575, 139)

$A'A =$ and $\parallel D'D$.

$\therefore \angle A'AB = \angle D'DC$. (498)

$\therefore$ face $AB' \cong DC'$, (147)

and face $AB' \parallel DC'$. (494)

591. Cor. *Any two opposite faces of a parallelopiped may be taken as bases.*

Ex. 1. In a rectangular parallelopiped, lettered as the above figure, find the length of the diagonal AC', if $AB = 9$, $BC = 12$, $CC' = 8$.

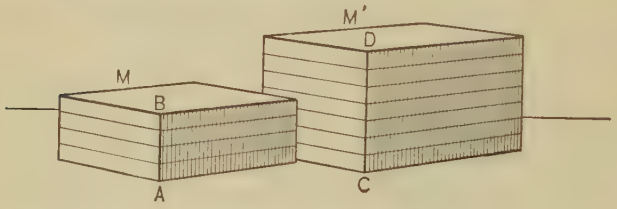
Ex. 2. In a right parallelopiped lettered similarly, find the length of the diagonal AC', if $AB = 5$, $BC = 3$, $CC' = 2$, and $\angle ABC = 120°$.

Ex. 3. Find the total area of a right prism if its altitude is 12 in., and its base is a triangle the sides of which are 13 in., 15 in., and 4 in. respectively.

Ex. 4. Find the total area of a right prism if its altitude is 9 in., and its base is a rhombus of side 4 in. and having an angle of 60°.

Proposition VI. Theorem

592. *Two rectangular parallelopipeds having equal bases are to each other as their altitudes.*



Given AB and CD, the altitudes of parallelopipeds M and M' which have equal bases.

To prove $M : M' = AB : CD$.

Case I. $\frac{AB}{CD}$ = a rational number.

Let $\frac{AB}{CD} = \frac{m}{n}$, where m and n are integers,

i.e. if CD is divided into n equal parts, and one of them is laid off on AB, AB contains m such parts.

Through the points of division pass planes parallel to the bases.

[To be completed by the student.]

Case II. $\frac{AB}{CD}$ is an irrational number.

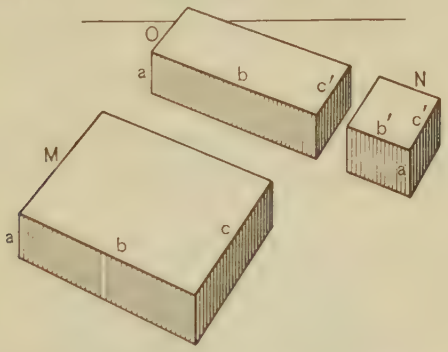
Use method of Prop. I, Bk. IV.

593. DEF. The **dimensions** of a rectangular parallelopiped are the three edges that meet at the same vertex.

594. COR. The preceding theorem may be stated: *Two rectangular parallelopipeds which have two dimensions in common are to each other as the third dimension.*

PROPOSITION VII. THEOREM

595. *Two rectangular parallelopipeds having equal altitudes are to each other as their bases.*



Given a, b, c and a, b', c', the three dimensions of rectangular parallelopipeds M and N respectively; a being the equal altitude.

To prove $M : N = b \times c : b' \times c'$.

Proof. Construct O a rectangular parallelopiped with dimensions a, b, and c'.

$$\frac{M}{O} = \frac{c}{c'}. \qquad (594)$$

$$\frac{O}{N} = \frac{b}{b'}. \qquad (594)$$

$$\therefore \frac{M}{N} = \frac{b \times c}{b' \times c'}. \qquad (\text{Ax. } 7)$$

596. Cor. *Two rectangular parallelopipeds that have one dimension in common are to each other as the products of their other two dimensions.*

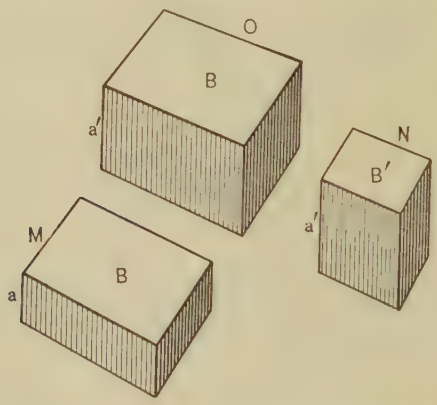
597. Note. The product of three lines, or the product of an area and a line, denotes the product of the numerical measures of these quantities.

Ex. 1. Two rectangular parallelopipeds have equal altitudes and bases whose dimensions are 4 and 7, and 5 and 9 respectively. Find the ratio of their volumes.

Ex. 2. What is the ratio of the volumes of two rectangular parallelopipeds having equal bases and altitudes a and b respectively?

Proposition VIII. Theorem

598. *Two rectangular parallelopipeds are to each other as the products of their bases and their altitudes.*



Given M and N, two rectangular parallelopipeds, B and B' their bases, and a and a' their altitudes respectively.

To prove $M : N = B \times a : B' \times a'$.

Proof. Construct O, a rectangular parallelopiped with base B, and altitude a'.

$$\frac{M}{O} = \frac{a}{a'}. \tag{592}$$

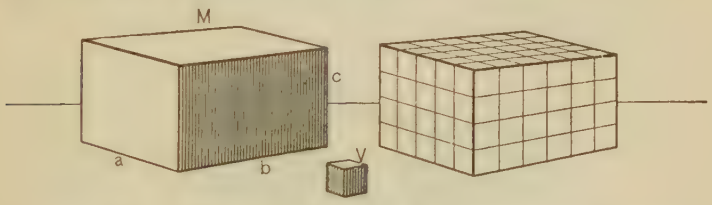
$$\frac{O}{N} = \frac{B}{B'}. \tag{595}$$

$$\therefore \frac{M}{N} = \frac{B \times a}{B' \times a'}. \tag{Ax. 7}$$

599. Cor. *Two rectangular parallelopipeds are to each other as the products of their three dimensions.*

PROPOSITION IX. THEOREM

600. *The volume of a rectangular parallelopiped is equal to the product of its three dimensions.*



Given a, b, c, the dimensions of a rectangular parallelopiped M.

To prove volume of $M = a \times b \times c$.

Hint. Construct V = unit of volume, and apply Prop. VIII.

601. Cor. 1. *The volume of a cube equals the cube of its edge.*

602. Cor. 2. *The volume of a rectangular parallelopiped equals the product of its altitude by its base.*

Ex. 1. Find the area of the base of a rectangular parallelopiped whose altitude is 6, if the solid is equivalent to a rectangular parallelopiped whose dimensions are 8, 12, and 15.

Ex. 2. Find volume of a rectangular parallelopiped the dimensions of whose base are 12 and 20, and the area of whose entire surface is 800.

Ex. 3. Find the volume of a rectangular parallelopiped the dimensions of whose base are 12 in. and a in., and the area of whose entire surface is $(120 + 34\,a)$ sq. in.

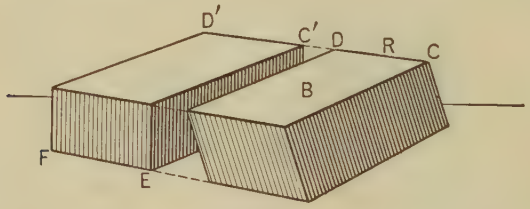
Ex. 4. Find the edge of a cube having a surface equal to the sum of the surfaces of two cubes whose edges are 120 in. and 209 in.

Ex. 5. The dimensions of a rectangular solid are proportional to 3, 4, and 5. Find the dimensions and the volume if the entire surface contains 2350 sq. in.

Ex. 6. Find the volume of a cube whose diagonal equals $\sqrt{18}$.

Proposition X. Theorem

603. *The volume of a right parallelopiped is equal to the product of one of its lateral faces and the corresponding altitude.*



Given R, the volume of a right parallelopiped, B one of its lateral faces, H the corresponding altitude.

To prove vol. $R = B \times H$.

Proof. Produce CD and the edges parallel to it. Take $C'D' = CD$, and pass planes $C'E$ and $D'F$ perpendicular to $C'D'$; thus forming parallelopiped $C'F$.

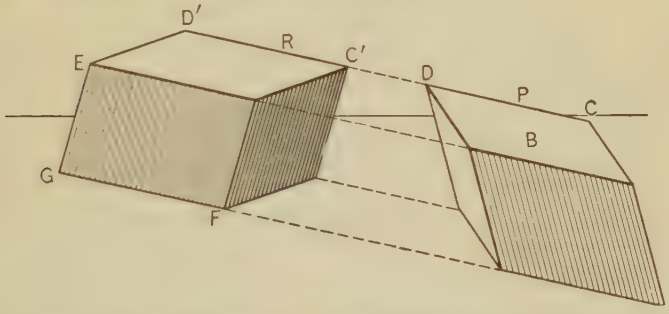
Then vol. $C'F$ = vol. R. (588)

But the three angles at C' are right angles. (540)

$\therefore$ $C'F$ is a rectangular parallelopiped. (577)
Hence $C'F$ = base $\times$ altitude. (602)
But the base of $C'F = B$, (354)
and the altitude of $C'F = H$. (519)
$\therefore R = B \times H$. (Sub.)

PROPOSITION XI. THEOREM

604. *The volume of any parallelopiped is equal to the product of its base and altitude.*



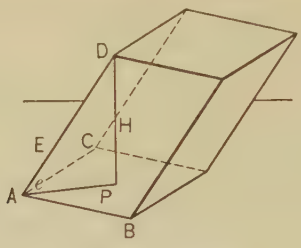
Given P, the volume of any parallelopiped, B its base, and H its altitude.

To prove vol. $P = B \times H$.

Proof. Produce edge CD and the edges parallel to it. Take $C'D' = CD$, and through C' and D' pass planes $C'F$ and $D'G$ perpendicular to $C'D'$; thus forming the parallelopiped R.

$R = P$. (588)
But R is a right parallelopiped. (576)
Hence $R = EC' \times$ its altitude. (603)
But $EC' = B$, (354)
and altitude of $R = H$. (519)
$\therefore R = B \times H$. (Sub.)
Hence $P = B \times H$.

605. NOTE. In any prism, the lateral edge E, the altitude H, and the projection of E upon the base form a right triangle. The angle e, opposite H, is the inclination of E to the base.



Ex. 1. Find the volume of a parallelopiped whose base has an area of 50 and whose lateral edge is 20, if the inclination of the lateral edge to the base is 30°.

Ex. 2. Find the volume of a parallelopiped the area of whose base is 20 and whose lateral edge is 10, if the inclination of the lateral edge to the base is 45°.

Ex. 3. In the preceding figure, find the volume of the parallelopiped if $AB = 6$, $AC = 5$, $\angle BAC = 30°$, $E = 10$, and the projection of E upon the base equals 8.

Ex. 4. In the same figure find the volume of the parallelopiped if $AB = 5$, $AC = 6$, $AD = 8$, $\angle BAC = 60°$, and the inclination of E to the base equals 60°.

Ex. 5. Find the volume of a parallelopiped the area of whose base is 10 square feet, if the projection of the lateral edge upon the base equals 2 feet, and the inclination of the lateral edge to the base is 60°.

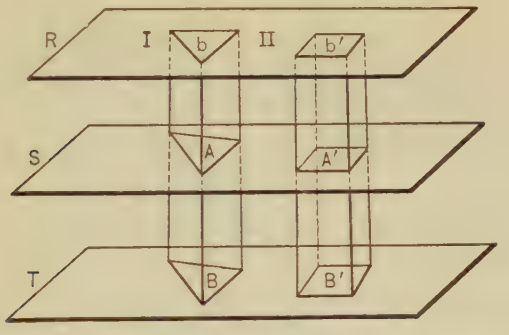
Ex. 6. A rectangular tank, 5 ft. long, 4 ft. wide, and 3 ft. deep, is to be lined with zinc, $\frac{1}{8}$ of an inch thick. How many cubic feet of zinc are required if 3 sq. ft. are allowed for overlapping?

Ex. 7. If one cubic inch of gold beaten into gold leaf will cover 20,000 sq. ft. of surface, what is the thickness of the gold leaf?

Ex. 8. An open tank is made of iron $\frac{1}{2}$ in. thick. The outer dimensions are as follows: length equals 2 ft., width 1 ft. 6 in., height 1 ft. If 1 cu. ft. of iron weighs 460 lb., find the weight of the tank.

Ex. 9. In a rainfall of $1\frac{1}{4}$ in., how many tons of water fall upon an acre of ground, if 1 cu. ft. of water weighs 62.5 lb.?

606. CAVALIERI'S THEOREM. *If two solids contained between the same two parallel planes have all their corresponding sections that are parallel to these planes equal, then the solids have equal volumes.*



Given the two volumes as above, and area A = area A' for all sections made by plane $S \parallel R$ and $\parallel T$.

Then the theorem asserts vol. I = vol. II.

The proof of this theorem can be made rigorous only by the application of the integral calculus.

607. THEOREM. *Two prisms having equivalent bases and equal altitudes are equivalent.*

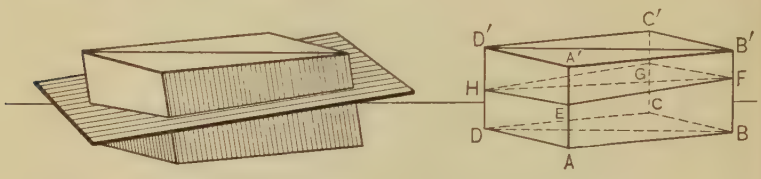
Using figure above, we have

area B = area B'	(Hyp.)
area A = area B	(579)
area A' = area B'	(579)
area A = area A'	(Sub.)
$\therefore$ vol. I = vol. II.	(606)

Ex. Apply this theorem to prove Prop. XI.

Proposition XII. Theorem

608. *The plane passed through two diagonally opposite edges of a parallelopiped divides it into two equivalent triangular prisms.*



Given $BDD'B'$, a plane passing through edges DD' and BB' of parallelopiped AC'.

To prove prism $ABD\text{–}A'$ = prism $BCD\text{–}C'$.

Proof. Draw right section $EFGH$, intersecting $DD'B'B$ in HF.

Plane $AB' \parallel$ plane DC'. (590)

$\therefore EF \parallel HG$. (486)

Similarly, $EH \parallel FG$.

$\therefore EFGH$ is a parallelogram.

$\triangle EFH \cong \triangle FGH$. (140)

$\therefore$ prism $ABD\text{–}A'$ = prism $BDC\text{–}C'$. (589)

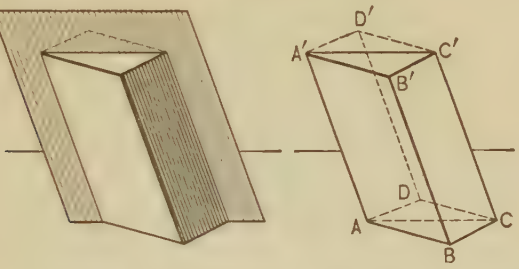
Ex. **1.** If in the figure of Prop. XII, $\triangle ABD = 60$ sq. in., and the altitude of the solid equals 10 in., find the volume of triangular prism $A'\text{–}ABD$.

Ex. **2.** In a right parallelopiped lettered as the figure above, $AB = 6$, $AD = 4$, $AA' = 8$, and $\angle DAB = 30°$. Find the volume of the triangular prism $A'\text{–}ABD$.

Ex. **3.** Prove Prop. XII, by applying Cavalieri's Theorem.

Proposition XIII. Theorem

609. *The volume of a triangular prism is equal to the product of its base by its altitude.*



Given V denoting the volume, B the base, and H the altitude of the triangular prism ABC–B'.

To prove vol. $V = B \times H$.

Proof. Upon the edges AB, BC, BB', construct the parallelopiped $ABCD$–D'.

$$ABC\text{–}B' = \tfrac{1}{2}\ ABCD\text{–}B'. \qquad (608)$$

The volume of $ABCD$–B' = base $ABCD \times H$. (604)

But $ABCD = 2\,B$.

Hence $ABCD\text{–}B' = 2\,B \times H$.

$$\therefore\ V = B \times H.$$

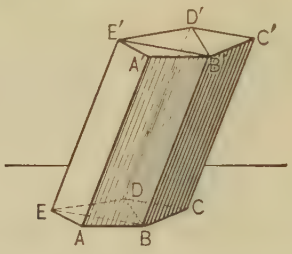
Ex. **1.** In the diagram for Prop. XIII, find the area of $\triangle ABC$, if the volume of ABC–B' is 200, and the altitude is 20.

Ex. **2.** In the same diagram find the volume of ABC–B' if $BA = 2$, $BC = 6$, $\angle ABC = 30°$, and $H = 7$.

Ex. **3.** In the same diagram, find the volume of ABC–B', if $AB = 4$, $BC = 6$, $BB' = 5$, $\angle ABC = 60°$, and the projection of BB' upon the base equals 4.

Proposition XIV. Theorem

610. *The volume of any prism is equal to the product of its base by its altitude.*



Given V denoting the volume, B the base, and H the altitude of any prism.

To prove $V = B \times H$.

Proof. Pass planes through one lateral edge BB', and the diagonals BE, BD, etc., of the base.

The prism will be divided into triangular prisms, each of which will have the altitude H.

$B'\text{–}ABE = H \times (ABE)$. (Why?)

$B'\text{–}BDE = H \times (BDE)$, etc.

$\therefore V = H \times (ABE + BDE + \cdots)$. (Why?)

Or $V = H \times B$.

611. Cor. 1. *Prisms that have equivalent bases are to each other as their altitudes.*

612. Cor. 2. *Prisms that have equal altitudes are to each other as their bases.*

613. Cor. 3. *Prisms that have equal altitudes and equivalent bases are equivalent.*

Ex. **1**. Find the volume of a regular quadrangular prism if the side of its base is 4, and the lateral edge is 6.

Ex. **2**. Find the volume of a regular hexagonal prism if the side of its base is 2, and the lateral edge is 5.

Ex. **3**. Find the altitude of a regular hexagonal prism, if the side of the base equals 4, and the volume equals $60\sqrt{3}$.

Ex. **4**. Find the volume of a regular triangular prism if the side of its base is 8, and the altitude is 10.

Ex. **5**. Find the volume of a right triangular prism, if the lateral edge is 9, and the sides of the base are 6, 8, and 10.

Ex. **6**. E is the lateral edge and $ABCD$ the base of a right quadrangular prism. Find the volume if $AB = 9$, $BC = 12$, $CD = 14$, $DA = 13$, $AC = 15$, and $E = 10$.

Ex. **7**. Find the volume of a triangular prism, if the sides of its base are 6, 4, and 4, the lateral edge $E = 8$, and the inclination of E to the base is 30°.

Ex. **8**. E is the lateral edge, p the projection of E upon the base, and s the side of the base of a prism. Find the volume, if $E = 7$, $p = 1$, $s = 1$, and the base is a regular hexagon.

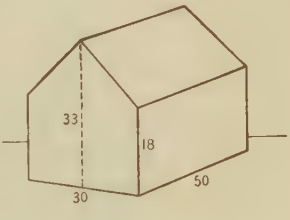
Ex. **9**. In the diagram for Prop. XIV, find AA', if $V = 200$, $ABCDE = 25$, and the projection of AA' upon the base is 6.

Ex. **10**. In the diagram for Prop. XIV, find the volume of the prism, if $ABCDE = 15$, the projection of AA' upon the base equals 3, and the inclination of AA' to the base equals 45°.

Ex. **11**. In the same diagram find the inclination of AA' to the base, if $AA' = 6$, $ABCDE = 20$, and $V = 60$.

Ex. **12**. *The diagonals of a parallelopiped bisect each other.*

Ex. **13**. Find the number of cubic feet of air in a building whose dimensions are given in the annexed diagram.



Ex. **14**. What is the total surface of the house described in the preceding exercise?

REVIEW EXERCISES

Ex. 1. The diagonal of a cube is $8\sqrt{3}$. Find the volume of the cube and its entire surface.

Ex. 2. The three lateral edges of a truncated right triangular prism are respectively 8″, 9″, and 12″; the sides of the base are respectively 12″, 27″, 25″. Find the lateral area.

Ex. 3. The base of a right prism is a rhombus, one side of which is 10 inches, and the shorter diagonal is 12 inches. The altitude of the prism is 15 inches. Find the volume.

Ex. 4. The base of a prism of altitude 7 inches is a regular hexagon whose side is 4 inches; the edges of the prism make angles of 60° with the altitude. Find the volume of the prism.

Ex. 5. Show that the volume of a cube equals the cube of the diagonal times $\frac{1}{9}\sqrt{3}$.

Ex. 6. Given two points A and B in two intersecting planes M and N respectively. To find Z in the line of intersection of M and N such that $AZ + ZB$ shall be a minimum.

Ex. 7. Three plane angles are 120°, 80°, 165°. Will these angles form a trihedral angle? Why?

Ex. 8. An isohedral trihedral is also isoangular, and conversely.

Ex. 9. If M and N be the feet of the perpendiculars to two planes from a point P, show that the line of intersection of the two planes is perpendicular to the plane MNP.

Ex. 10. Show that if the projections of two straight lines on the same plane are parallel, the lines are not necessarily parallel.

Ex. 11. Find the locus of those points of a plane at which a given straight line, not lying in the plane, subtends a right angle. Show when the locus becomes a point and when it disappears.

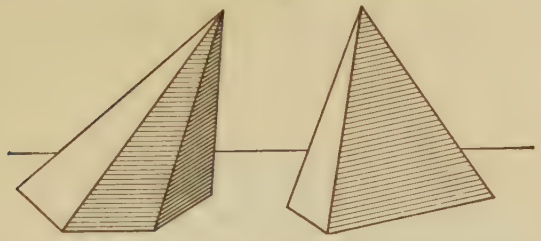
Ex. 12. The shortest distance between the diagonal of a cube and an edge which it does not meet is half the diagonal of one of the faces.

Ex. 13. If MA, MB, MC are the edges of a parallelopiped, and MD is a diagonal, then MD is divided by the plane of ABC in a trisection point.

Ex. 14. If the three pairs of opposite edges of a tetrahedron are equal, the faces are acute triangles.

Ex. 15. If the common section of a tetrahedron, and a plane is a parallelogram, the plane is parallel to a pair of opposite edges.

PYRAMIDS



614. DEF. A **pyramid** is a polyhedron bounded by the faces of a polyhedral angle and a plane cutting all the faces.

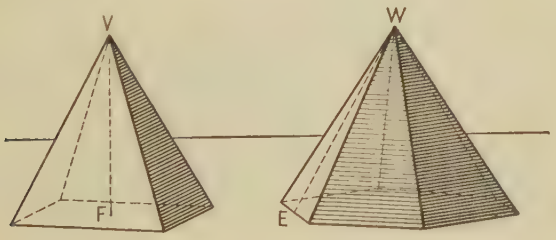
Obviously, all faces meeting at the vertex of the polyhedral angle are triangles, while the remaining surface is a polygon.

615. The **vertex of a pyramid** is the vertex of the polyhedral angle, the **lateral edges** are the edges of the polyhedral angle, the **lateral faces** are the triangles that meet at the vertex, the **lateral area** is the sum of the areas of the lateral faces, and the **base** is the face opposite the vertex.

616. DEF. A pyramid is **triangular**, **quadrangular**, etc., according as its base is a triangle, a quadrilateral, etc.

NOTE. A tetrahedron is a triangular pyramid.

617. DEF. The **altitude** of a pyramid is the length of the perpendicular from the vertex of the pyramid to the plane of the base, as *VF*.



618. DEF. A **regular** pyramid is a pyramid whose base is a regular polygon, the center of which coincides with the foot of the altitude.

A regular pyramid is also called a right pyramid.

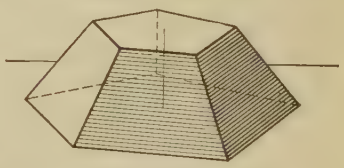
619. DEF. The **axis** of a regular pyramid is the line through the vertex and perpendicular to the base.

The lateral edges of a regular pyramid are equal since they cut off equal distances from the foot of the altitude.

Hence *the lateral faces of a regular pyramid are equal isosceles triangles.*

620. DEF. The **slant height of a regular pyramid** is the altitude of any one of the lateral faces, as *WE*. (Diagram on p. 375.)

621. DEF. A **truncated pyramid** is the portion of a pyramid included between the base and a section formed by a plane cutting all the lateral edges.



622. DEF. A **frustum of a pyramid** is a truncated pyramid in which the plane of the section is parallel to the base.

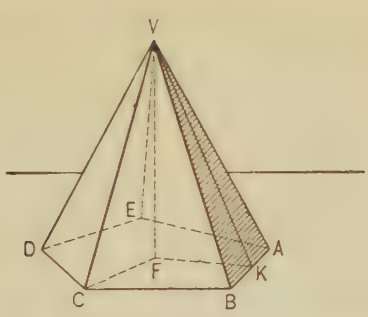
623. DEF. The **altitude of a frustum** is the perpendicular distance between its bases.

624. DEF. The **lateral area of the frustum** is the sum of the areas of its lateral faces; the **lateral faces of a frustum** of a regular pyramid are equal isosceles trapezoids.

625. DEF. The **slant height of a frustum** of a regular pyramid is the altitude of one of these trapezoids which form the lateral faces.

Proposition XV. Theorem

626. *The lateral area of a regular pyramid is equal to half the product of the slant height by the perimeter of the base.*



Given V–$ABCDE$, a regular pyramid of n lateral faces, P the perimeter of the base, L the lateral area, and S the slant height.

To prove $$L = \frac{P \times S}{2}.$$

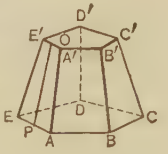
Proof. $\triangle VAB = \triangle VBC = \triangle VCD$, etc. (619)

$$\triangle VAB = \frac{AB \times S}{2}.$$

$$\therefore L = \frac{n \times AB \times S}{2},$$

or $$L = \frac{P \times S}{2}.$$

627. Cor. *The lateral area of a frustum of a regular pyramid is equal to one half the sum of the perimeters of the bases multiplied by the slant height.*



Hint. What is the shape of the lateral faces?

628. For many calculations relating to regular pyramids three types of right triangles are of importance. (See the figure for Prop. XV.) These types are illustrated by triangles VKF, VAK, and VFC. $\triangle VKF$ contains S, H, r, $\angle f$. $\triangle VAK$ contains E, S, $\frac{s}{2}$. $\triangle VFC$ contains E, H, R, $\angle e$.

The notations used here indicate:

E = lateral edge.
H = altitude of solid.
S = slant height.
s = side of base.
r = apothem of base.
R = radius of base.
$\angle e$ = inclination of E to base.
$\angle f$ = inclination of a lateral face to the base.

A few other notations used hereafter are:

B = area of base (or lower base if there are two).
b = area of the upper base.
L = lateral area.
V = volume.
T = total area.
h = altitude of lower triangular base.
h' = altitude of upper triangular base.
s' = side of upper base.
r' = apothem of upper base.
R' = radius of upper base.

Find lateral area of a:

Ex. 1. Regular triangular pyramid if $s = 6$, and $S = 4$.

Ex. 2. Regular quadrangular pyramid if $s = 8$, and $H = 3$.

Ex. 3. Regular octagonal pyramid if $E = 13$, and $s = 10$.

* Ex. 4. Regular quadrangular pyramid if $E = 8$, and $\angle e = 30°$.

Ex. 5. Find the total area of a regular triangular pyramid if $s = 2$, and $S = 3$.

Ex. 6. Find the total area of a regular hexagonal pyramid if $s = 4$, and $H = 2$.

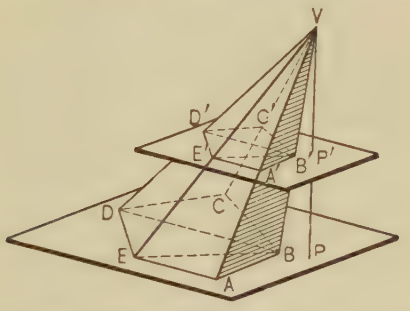
Ex. 7. Find the slant height of a regular pentagonal pyramid if $L = 90$, and $s = 3$.

Proposition XVI. Theorem

629. *If a pyramid be cut by a plane parallel to its base:*

(1) *The lateral edges and the altitude are divided proportionally.*

(2) *The section is a polygon similar to the base.*



Given pyramid $V\text{–}ABCDE$, cut by a plane parallel to its base, intersecting the lateral edges in A', B', C', D', E', and the altitude VP in P'.

1. **To prove** $\dfrac{VA'}{VA} = \dfrac{VB'}{VB} = \cdots \dfrac{VP'}{VP}$.

This follows directly from (495), after passing a plane through $V \parallel$ base.

2. **To prove** $ABCDE \sim A'B'C'D'E'$.

Proof. Pass a plane through VB and VE intersecting the parallel planes in BE and $B'E'$ respectively. In like manner obtain BD and $B'D'$.

$AB \parallel A'B'$, $BE \parallel B'E'$, and $EA \parallel E'A'$. (486)

$\therefore \angle ABE = \angle A'B'E'$, $\angle BEA = \angle B'E'A'$. (498)

$\therefore \triangle ABE \sim \triangle A'B'E'$. (305)

In like manner $\triangle BED \sim \triangle B'E'D'$, etc.

$\therefore ABCDE \sim A'B'C'D'E'$. (316)

630. Cor. 1. *A section of a pyramid parallel to the base is to the base as the square of its distance from the vertex is to the square of the altitude of the pyramid.*

It follows from the similarity of $\triangle VAB$ and $V'A'B'$ that

$$AB : A'B' = VA : VA' = VP : VP',$$

but

$$ABCDE : A'B'C'D'E' = \overline{AB}^2 : \overline{A'B'}^2 = \overline{VP}^2 : \overline{VP'}^2.$$

631. Cor. 2. *If two pyramids have equal altitudes and equivalent bases, sections made by planes parallel to the bases, and at equal distances from the vertices, are equivalent.*

Hint. If B is the area of the equal bases, and b and b' are the areas of the sections, then

$$B : b = \overline{VP}^2 : \overline{VP'}^2,$$

and

$$B : b' = \overline{VP}^2 : \overline{VP'}^2.$$ (Apply 275)

Ex. 1. The altitude of a regular triangular pyramid, each side of whose base is 4 ft., is 12 ft. Find the area of a section made by a plane parallel to the base and 4 ft. from the vertex.

Ex. 2. Each side of the base of a regular quadrangular pyramid is 3 ft. and its altitude is 8 ft. Find the area of a section made by a plane 3 ft. from the vertex and parallel to the base.

Ex. 3. The base of a regular hexagonal pyramid measures 10 in. on a side. If the altitude be 5 in., how far from the vertex must a plane be passed to form a section the area of which is $3\sqrt{3}$?

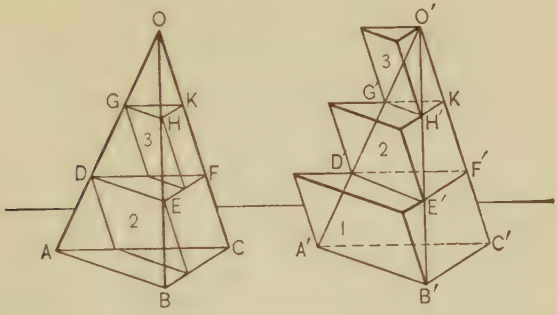
Ex. 4. Find the lateral area of a frustum of a regular triangular pyramid if $S = 6$, $s = 4$, and $s' = 3$.

Ex. 5. Find the lateral area of a frustum of a regular quadrangular pyramid if $s = 12$, $s' = 4$, and $H = 3$.

Ex. 6. The corner of a cube whose edge is 4″ is cut off by a plane passing through the mid-points of the edges which terminate at that vertex, and the process is repeated for each corner of the cube. Find the area of the surface of the solid that remains.

Proposition XVII. Theorem

632. *Two triangular pyramids which have equivalent bases and equal altitudes are equivalent.*



Given two triangular pyramids O–ABC and O'–$A'B'C'$ having equal altitudes, equivalent bases, and the volumes V and V' respectively.

To prove $V = V'$.

Proof. Suppose $V' > V$.

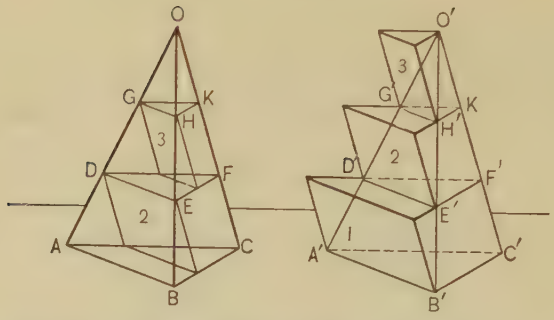
Place the pyramids so that their bases are in the same plane.

Divide the altitudes into n equal parts, calling the length of each part h.

Through the points of division pass planes parallel to the bases.

The corresponding sections thus formed are equivalent. (631)

On the base $A'B'C'$, and on each parallel section of O'–$A'B'C'$ as a lower base, construct a prism with lateral edges parallel to $O'C'$ and altitude equal to h. On each section of O–ABC as an upper base construct a prism with lateral edges parallel to OC and altitude equal to h.



Each prism of O–ABC is equivalent to the prism next above it in O'–$A'B'C'$, hence the *difference between the two sets of prisms* is the lowest prism in O'–$A'B'C'$. By increasing the number of parts into which each altitude is divided, each part h, and hence the volume of the lowest prism in O'–$A'B'C'$, can be diminished indefinitely. Or if P' and P denote the volumes of the two sets of prisms, the limit of $(P' - P) = 0$.

But $P' > V'$, (Ax. 11.)

and $P < V$. (Ax. 11.)

$\therefore P' - P > V' - V$.[1]

If, however, V' were greater than V, $V' - V$ would be a positive quantity, or a positive quantity would be less than a quantity which approaches zero as a limit, which is impossible.

$\therefore V'$ is not greater than V. In the same manner we may prove that pyramid V is not greater than pyramid V'.

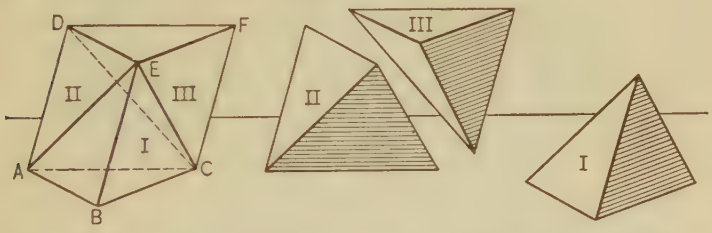
$\therefore V = V'$.

Ex. Prove this theorem by applying Cavalieri's Theorem.

[1] The principle involved here is an axiomatic one, although it can be derived from the other axioms, viz.: If unequals are subtracted from unequals of the opposite order, the remainders are unequal.

Proposition XVIII. Theorem

633. *The volume of a triangular pyramid is equal to one third of the product of its base by its altitude.*



Given V the volume, B the base, and H the altitude of the triangular pyramid E–ABC.

To prove $V = \frac{1}{3} B \times H$.

Proof. On ABC construct the prism ABC–DEF, having its lateral edges equal and parallel to EB.

The prism is composed of the pyramid E–ABC and the quadrangular pyramid E–$ACFD$.

Through ED and EC pass a plane cutting $ACFD$ in DC, forming two triangular pyramids, E–DCF and E–ADC.

Let us denote the volumes of E–ABC, E–ADC, and E–DCF by I, II, and III respectively.

Considering C as their common vertex, pyramids I and II have the same altitude and equivalent bases. (140)

$$\therefore \text{ I} = \text{II}. \qquad (632)$$

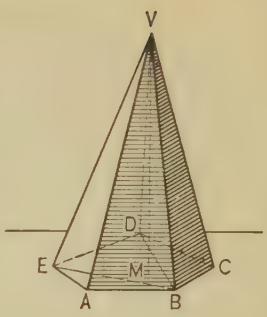
Considering E the vertex of II and III, it follows in like manner that $\text{II} = \text{III}$.

$$\therefore \text{ I} = \text{II} = \text{III}.$$

Hence, $\text{I} = \frac{1}{3}$ prism ABC–EFP.

Therefore $V = \frac{1}{3} B \times H$. (609)

634. Cor. 1. *The volume of any pyramid is equal to one third the product of its base by its altitude.*



For the pyramid may be divided into triangular pyramids by passing planes through any edge of the corresponding diagonals of the base, all having the common altitude VM, or H.

$$V\text{–}ABE = \tfrac{1}{3} H \times ABE.$$

$$V\text{–}BED = \tfrac{1}{3} H \times BED, \text{ etc.}$$

$$\therefore\ V\text{–}ABCDE = \tfrac{1}{3} H(ABE + BED + \cdots).$$

$$V\text{–}ABCDE = \tfrac{1}{3} H \times B.$$

635. Cor. 2. *The volumes of two pyramids are to each other as the products of their bases and altitudes.*

636. Cor. 3. *Pyramids having equivalent bases are to each other as their altitudes, and those having equal altitudes are to each other as their bases.*

637. Cor. 4. *Pyramids having equal altitudes and equivalent bases are equivalent.*

Ex. 1. Find the volume of a triangular pyramid, if its base equals 20, one of its lateral edges equals 12, and the inclination of this edge to the base is 30°.

Ex. 2. Find the volume of a triangular pyramid whose altitude equals 8, and the sides of whose base equal 14, 15, and 13.

Ex. 3. Find the altitude of a pyramid whose base is a square, if the volume equals 72, and the side of the base equals 6.

Ex. 4. Find the volume of a regular hexagonal pyramid if $s = 6$, and $H = 8$. (For notations see 628.)

Ex. 5. Find the volume of a regular triangular pyramid if $s = 4$, and $H = 6$.

Ex. 6. Find the volume of a regular quadrangular pyramid if $s = 6$, and $S = 5$.

Ex. 7. Find the volume of a regular quadrangular pyramid if $S = 12$, and $\angle f = 30°$.

Ex. 8. Find the volume of a regular hexagonal pyramid if $E = 12$, and $\angle e = 30°$.

Ex. 9. Find the volume of a regular quadrangular pyramid if $E = 11$, and $H = 7$.

Ex. 10. Find the volume of a regular hexagonal pyramid if $E = 7$, and $s = 1$.

Ex. 11. Find the volume of a regular quadrangular pyramid if $S = 4$, and $\angle f = 45°$.

Ex. 12. Find the lateral edge of a regular quadrangular pyramid if $V = 336$, and $H = 7$.

Ex. 13. The Great Pyramid (Pyramid of Ghizeh) is 233 m. square at the base and 146.5 m. high. Compute its volume, and its weight, if we assume the average weight of the material as 3 T. per cubic meter.

Ex. 14. What angle is formed by the diagonal of a cube and its projection on the base of the cube?

Ex. 15. A plane parallel to two opposite edges of a tetrahedron and intersecting the tetrahedron, cuts out a parallelogram.

Ex. 16. Prove, in a rectangular solid, that the four diagonals are equal to each other.

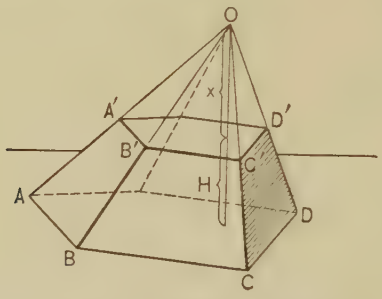
Ex. 17. The straight line joining each vertex of a tetrahedron with the intersection of the medians of the opposite face meet in a point that divides each line in two parts that have the ratio of 3 : 1.

Ex. 18. The lateral area of a pyramid is greater than the area of the base.

Ex. 19. The volume of a regular pyramid is equal to the product of its lateral surface and one third the distance from the center of the base to a lateral face.

Proposition XIX. Theorem

638. *If H denotes the altitude of a frustum of a pyramid, B the area of the lower base, and b the area of the upper base, the volume equals $\frac{1}{3}H(B+b+\sqrt{Bb})$.*



Proof. Produce the lateral edges of the frustum until they meet in O, and denote the altitude of the upper pyramid by x.

Obviously, $V = (O\text{–}ABCD) - (O\text{–}A'B'C'D')$

$$= \tfrac{1}{3}B(H+x) - \tfrac{1}{3}bx \qquad (633)$$

$$= \tfrac{1}{3}[BH + Bx - bx].$$

$$V = \tfrac{1}{3}[BH + x(B-b)].$$

To obtain the value of x, consider the proportion

$$B : b = (H+x)^2 : x^2. \qquad (630)$$

$$\therefore \sqrt{B} : \sqrt{b} = H + x : x. \qquad (287)$$

$$\therefore \sqrt{B} - \sqrt{b} : \sqrt{b} = H : x. \qquad (282)$$

$$\therefore x = \frac{H\sqrt{b}}{\sqrt{B} - \sqrt{b}}.$$

Substituting this value in $V = \frac{1}{3}[BH + x(B-b)]$,

$$V = \frac{1}{3}\left[BH + \frac{H\sqrt{b}(B-b)}{\sqrt{B} - \sqrt{b}}\right]$$

$$= \tfrac{1}{3}H[B + \sqrt{b}(\sqrt{B} + \sqrt{b})].$$

Or

$$V = \tfrac{1}{3}H[B + \sqrt{Bb} + b].$$

Ex. **1**. Find the volume of the frustum of a triangular pyramid, if the lower base is 9, the upper 4, and the altitude 5.

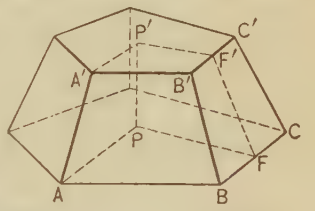
Ex. **2**. Find the volume of the frustum of a regular quadrangular pyramid, if the edge of the lower base is 7, of the upper 6, and if the altitude is 8.

Ex. **3**. The upper base of a frustum of a pyramid is 2, the lower 18. Find the altitude if the volume is 260.

Ex. **4**. The sides of the bases of a frustum of a regular triangular pyramid are 8 and 6, and the altitude is 9. Find the volume.

Ex. **5**. The sides of the bases of a frustum of a regular hexagonal pyramid are 4 and 2, and the altitude is 3. Find the volume.

639. NOTE. For more difficult frustum examples three trapezoids are of importance, viz. $APP'A'$, $FPP'F'$, and $BFF'B'$ (annexed figure).



Using the notations of (628),

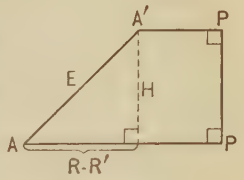
$APP'A'$ contains H, E, R, R', $\angle e$.

$FPP'F'$ contains H, S, r, r', $\angle f$.

$BFF'B'$ contains S, E, $\frac{s}{2}$, $\frac{s'}{2}$.

As each of these quadrilaterals contains two right angles, all parts can be found if three of them are given.

To solve a trapezoid we usually draw through one vertex a parallel to the opposite side and solve the resulting triangle.



Ex. **6**. Find the volume of the frustum of a regular hexagonal pyramid, if $s = 10$, $s' = 6$, and $E = 5$.

Ex. **7**. Find the volume of a frustum of a regular quadrangular pyramid if $S = 13$, $r = 11$, and $r' = 7$.

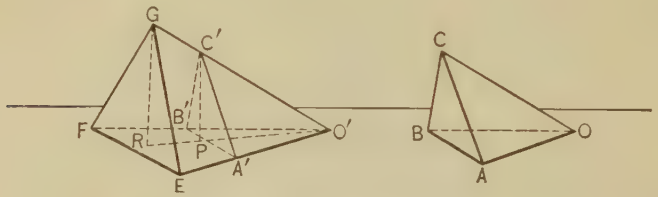
Ex. **8**. Find the lateral area of a frustum of a regular pentagonal pyramid if $s = 11$, $s' = 5$, and $E = 5$.

Ex. **9**. Find the lateral area of a frustum of a regular hexagonal pyramid if $s = 10$, $s' = 8$, $E = 4$.

Ex. **10**. Cleopatra's Needle is an Egyptian obelisk that was removed to Central Park, New York City. It is a frustum of a quadrangular pyramid 64 ft. high, 8 ft. square at the base, 5 ft. square at the top, and surmounted by a pyramid 7 ft. high. Find its volume and its weight. (Assume weight as 165 lb. per cubic foot.)

Proposition XX. Theorem

640. *The volumes of two triangular pyramids, that have a trihedral angle of one equal to a trihedral angle of the other, are to each other as the products of the three edges of these angles.*



Given V and V', volumes of the triangular pyramids O–ABC and O'–EFG, having the equal trihedral angles O and O'.

To prove $$\frac{V}{V'} = \frac{OA \times OB \times OC}{O'E \times O'F \times O'G}.$$

Proof. Place the pyramids so that trihedral $\angle O$ will coincide with trihedral $\angle O'$.

Draw $C'P$ and GR perpendicular to face $O'FE$, and let the plane of $C'P$ and GR intersect face $O'EF$ in line $O'PR$.

Since $O'A'B'$ and $O'EF$ are bases, and $C'P$ and GR the altitudes of pyramids C'–$O'A'B'$ and G–$O'EF$,

$$\frac{V}{V'} = \frac{\triangle O'A'B' \times C'P}{\triangle O'EF \times GR} = \frac{\triangle O'A'B'}{\triangle O'EF} \times \frac{C'P}{GR}. \qquad (635)$$

$$\frac{\triangle O'A'B'}{\triangle O'EF} = \frac{O'A' \times O'B'}{O'E \times O'F}. \qquad (377)$$

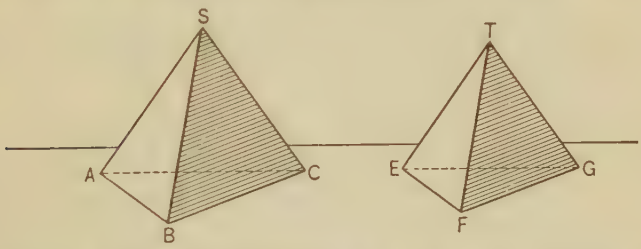
$$\frac{C'P}{GR} = \frac{O'C'}{O'G}. \qquad \text{(Why?)}$$

$$\therefore \frac{V}{V'} = \frac{O'A' \times O'B'}{O'E \times O'F} \times \frac{O'C'}{O'G} = \frac{OA \times OB \times OC}{O'E \times O'F \times O'G}. \qquad \text{(Why?)}$$

641. DEF. **Similar polyhedrons** are polyhedrons that have the same number of faces similar each to each and similarly placed, and have their corresponding polyhedral angles equal.

PROPOSITION XXI. THEOREM

642. *The volumes of two similar tetrahedrons are to each other as the cubes of their corresponding edges.*



Given V and V', the volumes of the similar tetrahedrons $S\text{–}ABC$ and $T\text{–}EFG$.

To prove $\dfrac{V}{V'} = \dfrac{\overline{SB}^3}{\overline{TF}^3} = \dfrac{\overline{SA}^3}{\overline{TE}^3}$, etc.

Proof. Trihedral $\angle S$ = trihedral $\angle T$. (641)

$$\frac{V}{V'} = \frac{SB \times SC \times SA}{TF \times TG \times TE} \text{ or } \frac{SB}{TF} \times \frac{SC}{TG} \times \frac{SA}{TE}. \quad (640)$$

But $$\frac{SB}{TF} = \frac{SC}{TG} = \frac{SA}{TE}. \quad \text{(Why?)}$$

$$\therefore \frac{V}{V'} = \frac{\overline{SB}^3}{\overline{TF}^3}. \quad \text{(Why?)}$$

Ex. 1. In the diagram for Prop. XXI, if $SA = 3$ and $TE = 2$, find the ratio of V to V'.

Ex. 2. In the same diagram find TF, if $SB = 2$, and $V : V' = 1 : 2$.,

REGULAR POLYHEDRONS

643. Def. A **regular polyhedron** is a polyhedron all of whose faces are equal regular polygons, and all of whose polyhedral angles are equal.

It is proved in Prop. XXII that only five regular polyhedrons are possible, viz. the tetrahedron, the octahedron, the cube, the icosahedron, and the dodecahedron. (For diagrams see pp. 354 and 391.)

Proposition XXII. Problem

644. *To determine the number of regular convex polyhedrons possible.*

A convex polyhedral angle must have at least three faces, and the sum of its face angles must be less than 360°. (558)

1. A convex polyhedral angle may be formed by combining three, four, or five equilateral triangles. Since each angle of an equilateral △ is 60°, the sum of six such angles is 360°, a sum greater than that of the face angles of a convex polyhedral angle.

Hence **three** regular convex polyhedrons are possible with **equilateral** ▲ as faces.

2. A convex polyhedral angle may be formed by combining three squares, but not by using four or more squares. (Why?)

Hence **one** regular convex polyhedron is possible with **squares** as faces.

3. A convex polyhedral angle may be formed by combining three regular pentagons, but not by using four, etc., pentagons. (Why?)

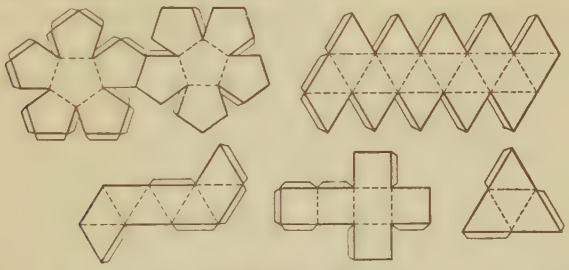
Hence **one** regular convex polyhedron can be formed having **regular pentagons** as faces.

4. It is impossible to form a convex polyhedral angle by combining hexagon, heptagon, octagon, etc. (Why?)

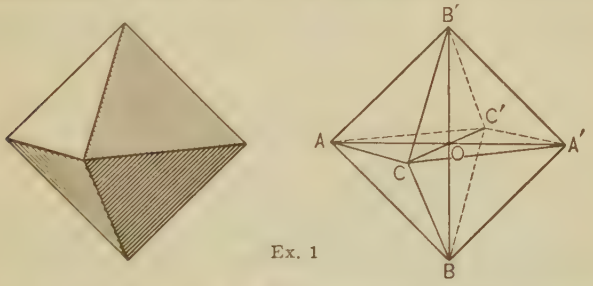
Hence only five regular convex polyhedrons are possible; the **tetahedron**, the **octahedron**, the **icosahedron** (having

equilateral triangles as faces), the **hexahedron** or the *cube* (having squares as faces), and the **dodecahedron** (having regular pentagons as faces).

645. To construct the regular polyhedrons, draw on stiff paper or cardboard the following diagrams. Cut partly through the paper along the dotted lines. Fold over and hold the edges in contact by pasting the narrow strips.



Ex. **1.** If three equal lines AA', BB', and CC' have their midpoints at a point O, and each line is perpendicular to the other two,



Ex. 1

then the points A, B, C, A', B', C', determine a regular octahedron. (The lines AA', BB', and CC' are called the *axes* of the octahedron.)

Ex. **2.** If an edge of an octahedron equals 2 in., find the length of the axes (AA', etc.).

Ex. **3.** If an edge of an octahedron equals 4 in., find the volume and the surface of the solid.

Ex. **4.** The three sides of the base of a pyramid are respectively 10, 17, and 21. Find the volume if the altitude is 5.

Ex. 5. The three sides of the base of a pyramid are respectively 9, 10, 17. Find the volume if a lateral edge is 20, and its projection upon the base equals 12.

Ex. 6. A lateral edge of a pyramid equals 10, and its inclination to the base is 30°. Find the area of the base if the volume of the pyramid is 100.

Ex. 7. The base of a pyramid is a rhombus whose diagonals are respectively 10 and 12. Find the volume if the altitude is 6.

Ex. 8. *The diagonals of a parallelopiped divide the figure into six equivalent pyramids.*

Ex. 9. If any point within a parallelopiped be joined to the 8 vertices, 6 pyramids are formed, of which the sum of any opposite two is equal to the sum of any other opposite two.

Ex. 10. Each edge of a triangular pyramid is equal to 10. Find the volume.

Ex. 11. The perimeter of the triangular base of a regular pyramid is 30. Find the volume if the altitude is 12.

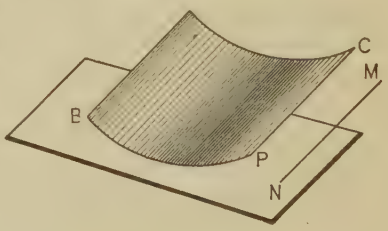
Ex. 12. The base of a pyramid is a parallelogram of base 10 and altitude 8. Find the volume if a lateral edge is equal to 6, and forms with the base an angle of 45°.

Ex. 13. The base of a pyramid is a rectangle having sides respectively equal to a and b. A lateral edge is equal to c, and is inclined to the base 30°. Find the volume.

Ex. 14. Find the altitude of a pyramid of base b, equivalent to another pyramid of base a and altitude h.

CYLINDERS

646. Def. A **cylindrical surface** is a surface generated by a moving straight line that continually intersects a fixed curve and is always parallel to a fixed straight line not in the same plane with the given curve.



647. Def. The **generatrix** of the surface is the moving straight line; the

directrix is the given curve; and an **element** of the surface is the moving line in any of its positions.

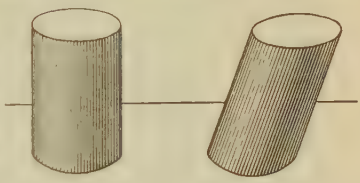
648. DEF. A **cylinder** is the figure formed by a cylindrical surface and two parallel planes cutting the elements of the cylindrical surface; the **bases** of a cylinder are the parts of the parallel planes bounded by the cylindrical surface and the **lateral surface** is that part of the cylindrical surface included between the planes.

The elements of a cylinder are equal since they are ‖ lines included between ‖ planes.

649. DEF. A **circular cylinder** is a cylinder whose bases are circles.

650. DEF. A **right cylinder** is a cylinder whose elements are perpendicular to the bases.

651. DEF. An **oblique cylinder** is one whose elements are oblique to the bases.

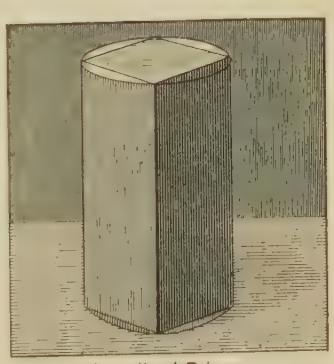


Right Cylinder Oblique Cylinder

652. DEF. The **altitude** of a cylinder is the perpendicular distance between the bases.

653. DEF. A straight **line** is **tangent** to a cylinder, if the line touches the lateral surface in one point but does not intersect it if produced. A **plane** is **tangent to a cylinder** if it contains one element of the cylinder and does not intersect the cylinder.

654. DEF. A prism is **inscribed** in a cylinder when its lateral edges are elements of the cylinder and its bases are inscribed in the bases of the cylinder.

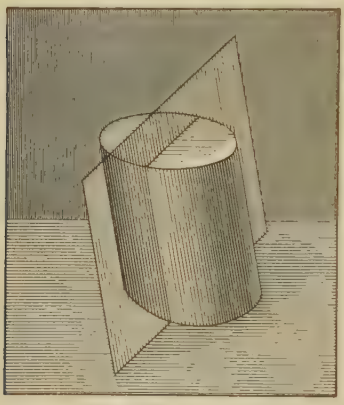


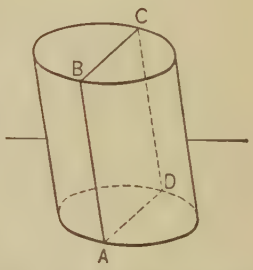
Inscribed Prism

655. Def. A **section of a cylinder** is the figure formed if the cylinder is intersected by a plane; a **right section** is a section formed by a plane perpendicular to the elements.

Proposition XXIII. Theorem

656. *Every section of a cylinder made by a plane passing through an element is a parallelogram.*





Given *ABCD*, a section of cylinder *AC*, made by plane through element *AB*.

To prove *ABCD* is a parallelogram.

Proof. Through *D*, in plane *AC*, draw a line $\parallel AB$.

This line is an element of the cylinder (646).

Since this line is in the plane *AC* and is an element of the cylindrical surface, it must be their intersection, and therefore coincides with *DC*.

$\therefore$ *DC* is a straight line $\parallel AB$.

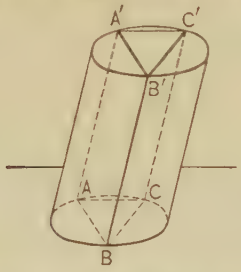
Also *AD* is a straight line $\parallel BC$. (Why?)

$\therefore$ *ABCD* is a parallelogram. (Why?)

657. Cor. *Every section of a right cylinder made by a plane passing through an element is a rectangle.*

PROPOSITION XXIV. THEOREM

658. *The bases of a cylinder are congruent.*



Given ABC and $A'B'C'$, the bases of the cylinder BC.

To prove base $ABC \cong$ base $A'B'C'$.

Proof. Let A and B be two fixed points in the lower base, and C any other point in the same base and AA', BB', CC' the corresponding elements.

Draw $\triangle ABC$ and $A'B'C'$.

AB', BC', and CA' are $\parallel$. (656)

Hence $AB = A'B'$, $BC = B'C'$, $CA = C'A'$.

$\therefore \triangle ABC \cong \triangle A'B'C'$. (s. s. s. = s. s. s.)

Place the upper base upon the lower base so that $A'B'$ coincides with AB. Then C' must coincide with C. But as C' is *any* other point in the perimeter of the upper base, all points in the upper base coincide with the corresponding points in the lower base and conversely.

$\therefore$ the bases are congruent.

659. COR. 1. *Any two parallel sections cutting all the elements of a cylinder are congruent.*

660. COR. 2. *Any section of a cylinder parallel to the base is congruent to the base.*

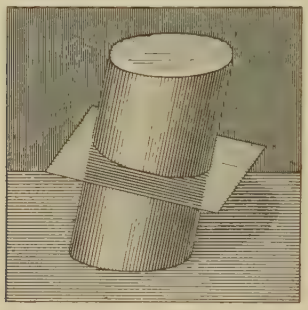
661. DEF. The **area of the lateral** surface of a cylinder is the limit which the lateral area of an inscribed prism ap-

proaches, if the number of its lateral faces is indefinitely increased.

The **volume of a cylinder** is the limit which the volume of the inscribed prism approaches if the number of its faces is increased indefinitely.

Proposition XXV. Theorem

662. *The lateral area of a circular cylinder is equal to the product of the perimeter of a right section of the cylinder by an element.*



Given L the lateral area, P the perimeter of a right section, and E an element of the cylinder AK; L' the lateral area, P' the perimeter of a section of a prism with a regular polygon as base, inscribed in cylinder AK.

To prove $L = P \times E$.

[1] **Proof.** The edge of the inscribed prism coincides with an element of the cylinder.

$$\therefore L' = P' \times E. \tag{581}$$

If the number of faces of the inscribed prism be increased, L' will approach L as a limit, and P' will approach P as a limit.

$$\therefore L = P \times E. \tag{414}$$

[1] See p. 410, sect. 701.

663. DEF. A **cylinder of revolution** is a right circular cylinder because it may be generated by a rectangle revolving about one of its sides as an **axis.**

664. DEF. **Similar cylinders of revolution** are cylinders generated by similar rectangles revolving about corresponding sides as axes.

665. COR. 1. *The lateral area of a cylinder of revolution is the product of the circumference of its base by its altitude.*

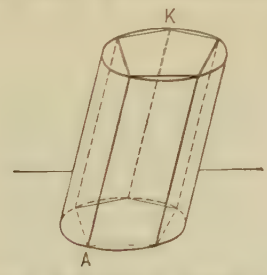
666. COR. 2. *If* L *denote the lateral area,* T *the total area,* H *the altitude, and* R *the radius of a cylinder of revolution,*

$$L = 2\pi RH, \text{ and } T = 2\pi R(H + R).$$

EX. Find the total area of a cylinder of revolution if $H = 6$, and $R = 2$.

PROPOSITION XXVI. THEOREM

667. *The volume of a circular cylinder is equal to the product of its base by its altitude.*



Given V the volume, B the base, and H the altitude, of cylinder AK; V' the volume, B' the regular polygon forming the base of a prism inscribed in AK.

[1] **To prove** $V = B \times H$.

HINT. Use the Theorem of Limits as in preceding proposition.[2]

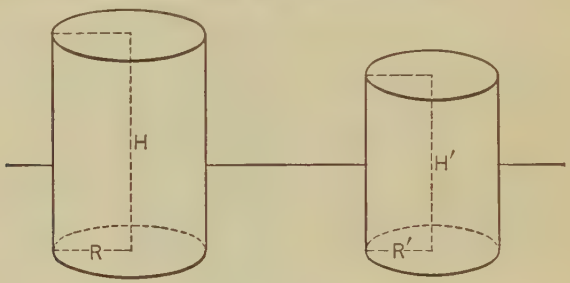
[1] See p. 410, sect. 701.

[2] Assume that B is the limit which B' approaches, if the number of sides is indefinitely increased.

668. Cor. *For a cylinder of revolution with radius of base* R,
$$V = \pi R^2 \times H.$$

Proposition XXVII. Theorem

669. *The lateral areas, or the total areas, of similar cylinders of revolution are to each other as the squares of their radii, or as the squares of their altitudes; and their volumes are to each other as the cubes of their radii, or as the cubes of their altitudes.*



Given L, L', the lateral areas; T, T', the total areas; V, V', the volumes; R, R', the radii; and H, H', the altitudes of two similar cylinders of revolution.

To prove $L : L' = T : T' = R^2 : R'^2 = H^2 : H'^2$,
and $V : V' = R^3 : R'^3 = H^3 : H'^3$.

Proof.
$$\frac{H}{H'} = \frac{R}{R'} = \frac{H + R}{H' + R'}. \tag{284}$$

$$\frac{L}{L'} = \frac{2\pi RH}{2\pi R'H'} = \frac{R}{R'} \times \frac{H}{H'} = \frac{R^2}{R'^2} = \frac{H^2}{H'^2}, \tag{666}$$

and
$$\frac{T}{T'} = \frac{2\pi R(H + R)}{2\pi R'(H' + R')} = \frac{R}{R'} \times \frac{H + R}{H' + R'} = \frac{R^2}{R'^2} = \frac{H^2}{H'^2}, \tag{666}$$

and
$$\frac{V}{V'} = \frac{\pi R^2 H}{\pi R'^2 H'} = \frac{R^2}{R'^2} \times \frac{H}{H'} = \frac{R^3}{R'^3} = \frac{H^3}{H'^3}. \tag{668}$$

Ex. 1. Find the ratio of the volumes of two similar cylinders of revolution, if their altitudes are to each other as 3 to 4.

Ex. 2. Find the radius of the base of a cylinder of revolution if the lateral surface equals 440, and the altitude equals 7 (assume $\pi = \frac{22}{7}$).

Ex. 3. Find the radius of the base of a cylinder of revolution if its lateral surface is equivalent to the sum of the lateral surfaces of three cylinders of revolution, whose radii are 3, 4, and 5, and all four cylinders have equal altitudes.

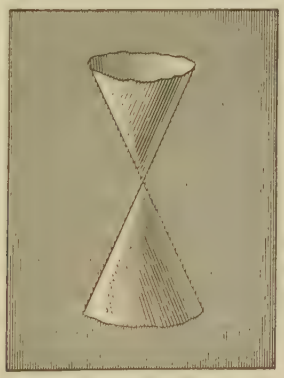
Ex. 4. Two cylinders of revolution have equal altitudes, and their radii are respectively 3 and 4. Find a third cylinder of revolution of the same altitude and equivalent to the sum of the two given cylinders.

Ex. 5. Find volume of a circular cylinder if the radius of the base is 3, an element, E, equals 4, and the inclination of E to the base is 45°.

CONES

670. DEF. A **conical surface** is a surface generated by a moving straight line which continually intersects a given fixed curve and constantly passes through a fixed point not in the same plane with the curve.

671. DEF. The **generatrix** of the surface is the moving straight line; the **directrix** is the given curve; and the **vertex** is the fixed point.



Conical Surface

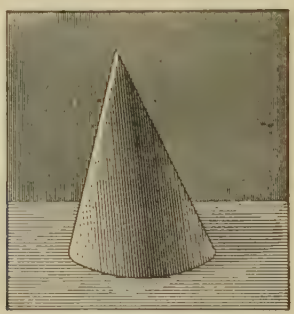
672. DEF. An **element** of a conical surface is the generatrix in any position.

673. DEF. The **upper and lower nappes** are the portions of the conical surface formed above and below the vertex when the length of the generatrix is unlimited.

674. DEF. A **cone** is the figure formed by a conical surface and a plane cutting all its elements.

675. DEF. The **lateral surface of the cone** is the conical surface; the **base of the cone** is the plane surface; the **vertex of the cone** is the vertex of the conical surface; and the **elements of the cone** are the elements of the conical surface.

676. DEF. A **circular cone** is a cone whose base is a circle; the **axis** of a circular cone is the straight line joining the vertex and the center of the base.



Circular Cone

677. DEF. A **right circular cone** is one whose axis is perpendicular to the base; an **oblique circular cone** is one whose axis is not perpendicular to the base.

NOTE. In this book only circular cones are treated.

678. DEF. The **altitude** of a cone is the perpendicular from the vertex to the plane of the base.

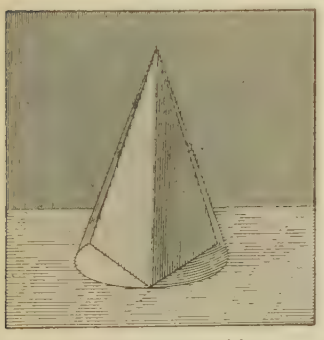
679. DEF. A **cone of revolution** is a right circular cone, for this latter may be generated by a right triangle revolving about one of its arms as an axis.

680. DEF. **Similar cones of revolution** are cones generated by the revolution of similar right triangles about corresponding arms.

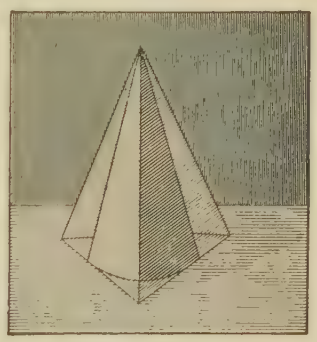
681. DEF. A line is **tangent** to a cone, if the line touches the cone in one point only and does not intersect it if produced.

682. DEF. A plane is **tangent** to a cone, if it contains an element but no other point of the cone.

683. DEF. A pyramid is **inscribed in a cone** when its base is inscribed in the base of the cone and its vertex coincides with the vertex of the cone.



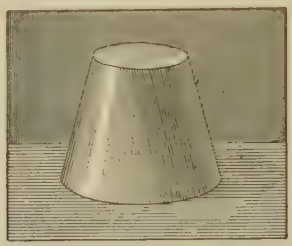
Inscribed Pyramid



Circumscribed Pyramid

684. A pyramid is **circumscribed about a cone** when its base is circumscribed about the base of the cone and its vertex coincides with the vertex of the cone.

685. DEF. A **frustum** of a cone is the portion included between its base and a plane parallel to the base. The **lower base** of the frustum is the base of the cone, and the **upper base** is the section made by the plane.



Frustum of a Cone

686. If pyramids are circumscribed about and inscribed in a cone with volume V, lateral surface L, base B, circumference of the base C, and the number of lateral faces of the pyramids is increased indefinitely, then

L is the limit of the lateral surfaces of the circumscribed pyramids.

V is the limit of the volumes of the inscribed pyramids.

B is the limit of the base of the inscribed pyramid.

C is the limit of the perimeter of the base of the circumscribed pyramid.

NOTE. The first two statements may be considered as definitions; the last two are assumptions.

It will be further assumed in the following section that

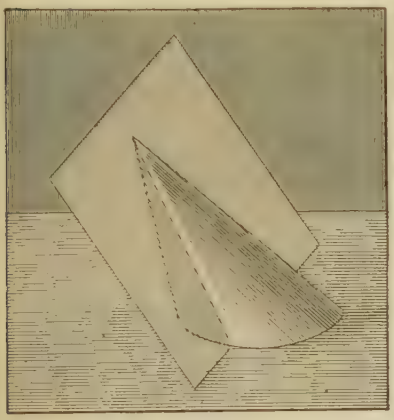
(*a*) The limit of the sum of several variables equals the sum of the limits of these variables.

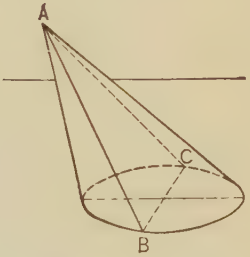
(*b*) The limit of the product of two variables equals the product of their respective limits, when neither of these limits is zero.

All four assumptions are capable of proof based upon proper definitions.

PROPOSITION XXVIII. THEOREM

687. *Every section of a cone made by a plane passing through its vertex is a triangle.*





Given ABC, a section of the cone made by plane passing through vertex A.

To prove ABC a triangle.

Proof. Join A and B by a straight line.

This line is an element of the cone. (670)

This line lies in the given plane. (481)

$\therefore$ this line is the intersection of the conical surface and the given plane.

In like manner it follows that the straight line AC is the intersection of conical surface and the given plane.

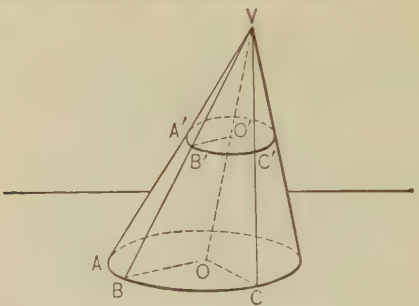
BC is a straight line. (485)

$\therefore$ ABC, the section, is a triangle.

688. COR. *Every section of a right cone made by a plane passing through its vertex is an isosceles triangle.*

Proposition XXIX. Theorem

689. *Every section of a circular cone made by a plane parallel to the base is a circle.*



Given $A'B'C'$, a section of cone $V\text{–}ABC$ made by a plane $\parallel ABC$, O the center of the base, and O', the point of intersection of VO and plane $A'B'C'$.

To prove. $A'B'C'$ is a circle.

Proof. Pass planes through VO and OB, a fixed radius, and through VO and OC, a variable radius, and let them intersect plane $A'B'C'$ in $O'B'$ and $O'C'$ respectively.

$$OB \parallel O'B'. \tag{486}$$

$$\therefore \triangle VOB \sim \triangle VO'B'. \tag{307}$$

$$\therefore \frac{VO}{VO'} = \frac{OB}{O'B'}. \tag{303}$$

Similarly $$\frac{VO}{VO'} = \frac{OC}{O'C'}. \tag{303}$$

But $$OB = OC. \tag{Why?}$$

$$\therefore O'B' = O'C'. \tag{275}$$

$\therefore A'B'C'$ is a circle.

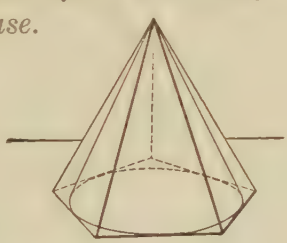
690. Cor. 1. *The axis of a circular cone passes through the center of every section which is parallel to the base, or*

The locus of the centers of the sections of a circular cone made by planes parallel to the base is the axis of the cone.

691. Cor. 2. *Sections made by planes parallel to the bases of a circular cone are to each other as the squares of their radii, or as the squares of their distances from the vertex of the cone.*

Proposition XXX. Theorem

692. *The lateral area of a cone of revolution is equal to half the product of the slant height by the circumference of the base.*



Given L lateral area, C the circumference of the base, and S the slant height of the cone; L' the lateral area, P the perimeter of the regular polygon forming the base of a circumscribed pyramid.

[1] **To prove** $L = \frac{1}{2} C \times S$.

Hint. Circumscribe a pyramid; its slant height is S. Use Theorem of Limits.

693. Cor. If L is the lateral area, T the total area, H the altitude, S the slant height, R the radius of the base, of a cone of revolution,

$$L = \pi R \cdot S.$$

$$T = \pi R(S + R).$$

Ex. 1. Find the lateral area of a cone of revolution whose radius is 12 and whose altitude is 5.

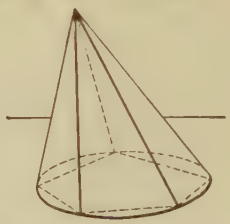
Ex. 2. Find the lateral area of a cone of revolution if the hypotenuse of the generating triangle is 10 in. and the acute angles are 45° each.

Ex. 3. Find the altitude of a cone of revolution if $L = 36\pi$, and $R = 4$.

[1] See p. 410, sect. 701.

Proposition XXXI. Theorem

694. *The volume of a cone is equal to one third the product of its base by its altitude.*



Given V the volume, B the base, and H the altitude of the cone; V' the volume, B' the regular polygon forming the base of an inscribed pyramid.

[1] **To prove** $V = \frac{1}{3} B \times H.$

Hint. Use Theorem of Limits. $V' = \frac{1}{3} B' \times H.$ (633)

695. Cor. *If the cone is a cone of revolution, with R as radius of B, then* $V = \frac{1}{3} \pi R^2 H.$

Ex. 1. Find the volume of a cone of revolution with radius 6 and altitude 2.

Ex. 2. Find the volume of a cone of revolution whose radius is 5 and whose slant height is 13.

Ex. 3. Find the lateral surface of a cone of revolution if its volume is 314 and its altitude is 3. (Assume $\pi = 3.14$.)

Ex. 4. The axis of a circular cone equals 17 in., its projection upon the base equals 8 in. Find radius of the base if the volume equals $80\,\pi$.

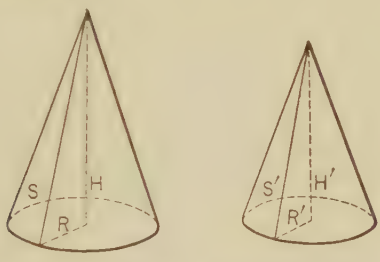
Ex. 5. A cone of revolution 3 in. high with a base whose area is $81\,\pi$ sq. in. is cut by a plane parallel to the base and 2 in. from it. Find the volume of the frustum.

Ex. 6. A cone of wood has its vertical angle equal to 60°, and the radius of its base equal to 2 in. A cylindrical hole of one-inch radius is bored through the entire cone, the axis of the hole coinciding with that of the cone. How much of the cone goes into chips?

[1] See p. 410, sect. 701.

Proposition XXXII. Theorem

696. *The lateral areas, or the total areas, of two similar cones of revolution are to each other as the squares of their altitudes, as the squares of their radii, or as the squares of their slant heights; and their volumes are to each other as the cubes of their altitudes, as the cubes of their radii, or as the cubes of their slant heights.*



Given L, L' the lateral areas, T, T' the total areas, V, V' the volumes, H, H' the altitudes, R, R' the radii, S, S' the slant heights of two similar cones of revolution.

To prove $L : L' = T : T' = H^2 : H'^2 = R^2 : R'^2 = S^2 : S'^2$,

and $V : V' = H^3 : H'^3 = R^3 : R'^3 = S^3 : S'^3$.

The proof is similar to that of Prop. XXVII.

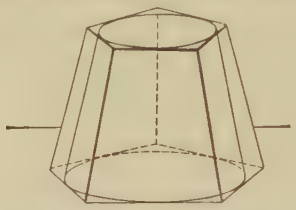
Ex. **1.** Find the ratio of the total areas of two similar cones of revolution whose altitudes are (*a*) 12 in. and 24 in. respectively; (*b*) 5 in. and *d* in. respectively.

Ex. **2.** The volume of a cone of revolution is 343 cu. in. and its altitude is 7 in. Find volume of a similar cone of altitude (*a*) 8 in.; (*b*) 15 in.

Ex. **3.** Find the volume of the solid generated by revolving a right triangle about its hypotenuse as an axis, if the legs are 15″ and 20″ respectively.

Proposition XXXIII. Theorem

697. *The lateral area of a frustum of a cone of revolution is equal to one half the sum of the circumferences of its bases multiplied by its slant height.*



Given L the lateral area, C and C' the circumferences of the bases, R and R' their radii, and S the slant height of the frustum; L' the lateral area, P and P' the perimeters of the bases, of a frustum of a regular pyramid circumscribed about the cone.

To prove $L = \frac{1}{2}(C + C') \times S.$

[1] **Proof.** The slant height of the frustum of the circumscribed pyramid $= S.$ (Why?)

$\therefore L' = \frac{1}{2}(P + P') \times S.$ (Why?)

[To be completed by the student. Use Theorem of Limits.]

698. Cor. *The lateral area of a frustum of a cone of revolution is equal to the circumference of a section* ***equidistant*** *from its bases multiplied by its slant height.*

For $\frac{1}{2}(C + C') = \frac{1}{2}(2\pi R + 2\pi R') = 2\pi\left(\frac{R + R'}{2}\right).$

But $2\pi\left(\frac{R + R}{2}\right)$ = the circumference of a section equidistant from the bases.

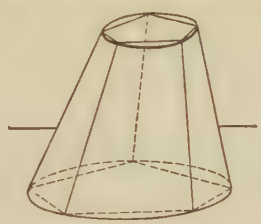
Ex. Find the lateral area of the frustum of a cone of revolution, if the radii of the bases are respectively 6 and 7, and

(*a*) the slant height equals 3. (*b*) the altitude equals 12.

[1] See p. 410, sect. 701, note.

PROPOSITION XXXIV. THEOREM

699. *The volume of a frustum of a circular cone is equivalent to the sum of its bases, and the mean proportional between its bases, multiplied by one third its altitude.*



Given V the volume, B the lower base, b the upper base, H the altitude of the frustum; V' the volume, B' the lower base, b' the upper base, of an inscribed frustum of a pyramid with regular polygons as bases.

To prove $V = \frac{1}{3} H(B + b + \sqrt{B \times b})$.

[1] **Proof.** The altitude of the frustum of the pyramid $= H$.

Then $V' = \frac{1}{3} H(B' + b' + \sqrt{B' \times b'})$. (638)

By increasing the number of the lateral faces of the inscribed frustum indefinitely,

V' becomes a variable with V as a limit,

B' approaches B as a limit,

b' approaches b as a limit, and $B' \times b'$, $B \times b$ as a limit.

$\therefore B' + b' + \sqrt{B' \times b'}$ approaches

$B + b + \sqrt{B \times b}$ as a limit.

$\therefore V = \frac{1}{3} H(B + b + \sqrt{B \times b})$. (Why?)

700. COR. *If the frustum is that of a cone of revolution, with R and R' as radii of its bases,*

$$V = \tfrac{1}{3} \pi H(R^2 + R'^2 + RR').$$

[1] See p. 410, sect. 701, note.

NOTE. The preceding proposition may be stated:

The volume of a frustum of a circular cone is equivalent to the sum of the volumes of three cones whose common altitude is the altitude of the frustum, and whose bases are the lower base, the upper base, and the mean proportional between the bases of the frustum. For, the value of V may be written,

$$\tfrac{1}{3} H \times B + \tfrac{1}{3} H \times b + \tfrac{1}{3} H \times \sqrt{B \times b}.$$

Ex. 1. Find the volume of a frustum of a circular cone, if $R = 5$, $R' = 4$, and $H = 6$.

Ex. 2. The masonry of a Dutch windmill forms a hollow frustum of a cone, 50 ft. high. The outer diameters are 35 ft. at the bottom and 30 ft. at the top. The inner diameters are 30 ft. at the bottom and 26 ft. at the top. How many cubic feet of stone are in the masonry of the windmill?

Ex. 3. The radii of the upper and lower bases of the frustum of a right circular cone are 3 in. and 9 in. respectively; the altitude of the frustum is 8 in. Find (1) the slant height, (2) the volume, (3) the total surface.

Ex. 4. What is the altitude, to the nearest tenth of an inch, of the frustum of a right circular cone whose bases have radii of 3 in. and 5 in. respectively and whose volume equals that of a circular cone with an altitude of 6 in. and a base with a radius of 4 in.?

Ex. 5. Find the lateral surface of the frustum in Ex. 1.

Ex. 6. An isosceles trapezoid revolves completely about the line that joins the centers of its two bases. The lengths of the two bases are 6 in. and 12 in. respectively, and the volume generated is $105\,\pi$ cu. in. Find the altitude of the trapezoid.

701. *Any property of a prism (or pyramid) with a regular polygon as a base, that is independent of the number of sides of the base, is a property of circular cylinders (or circular cones).*

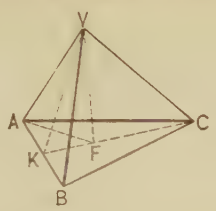
This statement can be proved by more advanced methods and can be used in proving Props. XXV, XXVI, XXX, and XXXI in Book VII.

Ex. 1. Prove Props. XXV, XXVI, XXX, and XXXI in Book VII, by applying the assumption made in section 701.

Ex. 2. Replacing, in 701, "pyramid" by "frustum of a pyramid," prove Props. XXXIII and XXXIV.

MISCELLANEOUS EXERCISES

Ex. 1. Find the lateral edge, lateral area, and volume of a regular triangular pyramid each side of whose base is 5 and whose altitude is 3.[1]



$AF = \frac{2}{3}h = \frac{2}{3}(\frac{5}{2}\sqrt{3}) = \frac{5}{3}\sqrt{3}$; $KF = \frac{1}{3}h = \frac{5}{6}\sqrt{3}$.

$E = VA = \sqrt{H^2 + \overline{AF}^2} = \sqrt{9 + \frac{75}{9}} = \sqrt{\frac{156}{9}} = \frac{2}{3}\sqrt{39}$.

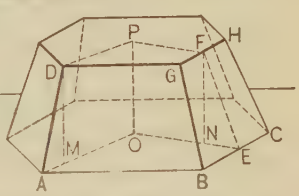
$S = VK = \sqrt{H^2 + \overline{KF}^2} = \sqrt{9 + \frac{75}{36}} = \sqrt{\frac{399}{36}} = \frac{1}{6}\sqrt{399}$.

$L = \frac{1}{2}S(AB + BC + CA) = \frac{1}{2} \cdot \frac{1}{6}\sqrt{399} \cdot 15 = \frac{5}{4}\sqrt{399}$.

$V = \frac{1}{3}B \times H = \frac{1}{3}(\frac{25}{4}\sqrt{3}) \cdot 3 = \frac{25}{4}\sqrt{3}$.

Ex. 2. Find the lateral edge, lateral area, and volume of a frustum of a regular hexagonal pyramid, the sides of whose bases are 14 and 6, respectively, and whose altitude is 8.

Let O and P be the centers of the lower and upper bases. Draw OE and PF perpendicular to BC and GH, respectively. Draw OA, PD, $DM \perp OA$, and $FN \perp OE$.



$NE = OE - PF = 7\sqrt{3} - 3\sqrt{3} = 4\sqrt{3}$.

$S = FE = \sqrt{\overline{FN}^2 + \overline{NE}^2} = \sqrt{64 + 48} = 4\sqrt{7}$.

$AD = \sqrt{\overline{DM}^2 + \overline{AM}^2} = \sqrt{64 + 64} = 8\sqrt{2}$.

$L = \frac{6}{2}[(14 + 6)4\sqrt{7}] = 240\sqrt{7}$.

$V = (B + b + \sqrt{Bb})\frac{H}{3} = (294\sqrt{3} + 54\sqrt{3} + 126\sqrt{3})\frac{8}{3} = 1264\sqrt{3}$.

Ex. 3. Let s = a side of the base, n the number of sides of the base, and H the altitude of a regular pyramid. Find the lateral edge, and the volume if

(a) $n = 3$, $s = 12$, $H = 12$.

(b) $n = 4$, $s = 6$, $H = 9$.

[1] For notations see (628). For the principle underlying the solutions of this and the following examples see (628) and (639).

Ex. 4. Using the notations of the preceding exercise find the lateral area if

(*a*) $s = 6,\ n = 3,\ H = 1.$

(*b*) $s = 8,\ n = 4,\ H = 5.$

Ex. 5. Let s = a side of the lower base, s' a side of the upper base, n the number of sides of the base, and H the altitude of a frustum. Find the lateral edge, and the volume if

(*a*) $s = 10,\ s' = 6,\ n = 4,\ H = 1.$

(*b*) $s = 8,\ s' = 5,\ n = 4,\ H = 3.$

(*c*) $s = 10,\ s' = 6,\ n = 6,\ H = 2.$

Ex. 6. Using the notations of the preceding exercise find the lateral area if

(*a*) $s = 10,\ s' = 2,\ n = 4,\ H = 5.$

(*b*) $s = 6,\ s' = 4,\ n = 4,\ H = 3.$

Ex. 7. Find V of a regular pyramid, if $E = 11,\ s = 12$, and $n = 4$.

Ex. 8. The total surface of a cube is 336 sq. in. Find its volume.

Ex. 9. The lateral area of a right prism is 140, and its base is a triangle whose sides are 5, 7, and 8. Find its volume.

Ex. 10. Find the volume of an oblique prism whose base is a regular hexagon inscribed in a circle of radius 5 in., if a lateral edge be 8 in. and its projection on the plane of the base be 3 in.

Ex. 11. The base of a pyramid whose altitude is 24 is a square of side 10. Find the area of a section 4 in. from the vertex and parallel to the base.

Ex. 12. How far from the vertex, on the lateral edge a, must a plane $\parallel$ to the base be passed to divide a pyramid into two equivalent parts?

Ex. 13. The altitude of a pyramid, having for its base a regular hexagon, each of whose sides is 8 in., is 28 in. Find the distance from the vertex of a section whose area is $\frac{3\sqrt{3}}{2}$ sq. in.

Ex. 14. Find the surface and volume of a cube in which the diagonal of each face is 15 in.

Ex. 15. Find the surface and volume of a cube whose diagonal is 30 in.

Ex. 16. The base of a right prism is a quadrilateral $ABCD$, in which $AB = 7$ in., $BC = 20$ in., $CD = 15$ in., $DA = 24$ in., and the angles at A and C are right angles. If the height of the prism is 12 in., find its entire surface and volume.

Ex. 17. Find the lateral area of a right pyramid having the same base and height as a cube whose edge is 10 in.

Ex. 18. *The diagonals of a rectangular parallelopiped are equal.*

Ex. 19. Find the lateral area of the frustum of a regular pyramid the altitude of which is 15 in., and the bases of which are squares of sides 40 in. and 24 in., respectively.

Ex. 20. A cone of revolution 12 in. in height and 16 in. in diameter at the base is cut by a plane parallel to the base and 9 in. from it. Find the lateral area and volume of the frustum so formed.

Ex. 21. The mid-points of two pairs of opposite edges of a tetrahedron determine a parallelogram.

Ex. 22. The altitude of a right prism is 10 in., and the base is a regular hexagon whose sides are 8 in. Find the surface and volume of the greatest possible cylinder, of the same axis, that can be cut from the prism.

Ex. 23. The base of a right pyramid is a regular hexagon whose sides are 20 in., and the lateral faces are inclined to the base at an angle of 60°. Find the volume.

Ex. 24. Lines joining the mid-points of opposite edges of a tetrahedron meet in a point and bisect each other.

Ex. 25. The altitude of a cone of revolution is 27 in., and its curved surface is 7 times the area of its base. Find the radius of the base.

Ex. 26. Find the volume of a regular hexagonal pyramid, if the lateral edge equals 13, and the side of the base equals 11.

Ex. 27. Find the volume and surface of a regular tetrahedron whose edge equals 4 in.

Ex. 28. Find the total surface of a regular tetrahedron, if the perpendicular from one vertex to the opposite face is 5 in.

Ex. 29. The corners of a cube are cut off by planes which pass through the middle points of each set of edges meeting at a common vertex. If the edge of the cube is 2 ft., find the volume of the solid formed.

Ex. **30**. A cylindrical vessel 10 in. in diameter is required to hold 10 gal. Find the height to the nearest tenth of an inch, if one gallon equals 231 cu. in.

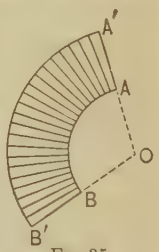
Ex. **31**. Find the weight of a hollow cylindrical column 5 ft. long, 1 ft. in outer diameter, and 1 in. thick, if the material is $7\frac{1}{2}$ times as heavy as water (*i.e.* the specific gravity = 7.5), and 1 cu. ft. of water weighs 62.5 lb.

Ex. **32**. A cylindrical vessel, 6 in. in diameter, is partly filled with water. Upon immersing a body of irregular shape, the surface of the water rises 2 in. Find the volume of the body.

Ex. **33**. How many yards of platinum wire $\frac{1}{100}$ in. thick can be made from a cubic inch of platinum?

Ex. **34**. By folding a semicircular piece of paper whose radius is 5 in. a conical surface may be formed. Find the volume and the total surface of the cone determined by this surface.

Ex. **35**. The area $AA'B'B$, bounded by two concentric arcs and parts of two radii is folded so as to form the lateral surface of a frustum of a cone whose lower base equals 36π. Find $\angle O$, and the volume of the frustum if $OA = 5$ and $OA' = 10$.



Ex. 35

Ex. **36**. By folding a piece of paper similar to the figure of the preceding exercise we wish to obtain a frustum of a cone, whose slant height equals 5, and whose bases have the radii 9 and 6 respectively. Find OA, OA', and $\angle O$.

Ex. **37**. A cylindrical tin can closed at the top requires the least amount of tin if its height equals its diameter. Find the diameter of such a can which holds one quart. (1 qt. = 57.75 cu. in.)

Ex. **38**. How many square feet of ground are covered by a conical tent 8 ft. high, if the canvas contains $188\frac{4}{7}$ sq. ft.? (Assume $\pi = 3\frac{1}{7}$.)

Ex. **39**. A conical vessel whose height equals the diameter of the base contains water to a depth of 6 in. Upon immersing a cube, the water rises 2 in. Find the edge of the cube.

Ex. **40**. A cylindrical boiler which is partly filled with water has a length of 6 ft. and a diameter of 2 ft. Find the volume of the water if its greatest depth is 6 in. and the axis of the cylinder has a horizontal position.

REVIEW EXERCISES

Ex. 1. Given e the length of an edge of a regular tetrahedron, show that the total surface $= e^2\sqrt{3}$, and vol. $= \frac{e^3}{12}\sqrt{2}$.

Ex. 2. Find the volume and the total surface of a regular octahedron in terms of its edge e.

Ex. 3. Compare the total surfaces, also the volumes, of two right circular cylinders generated by revolving rectangles, having bases a and b, and altitudes b and a respectively, about their altitudes as axes.

Ex. 4. Compare the total surfaces and volumes obtained by revolving a right triangle about each arm as an axis.

Ex. 5. Let AB and CD be two lines not in the same plane (skew lines), and let E be any point on AB and F any point on CD. What is the locus of the mid-point of line EF? Give reasons.

Ex. 6. A regular hexagon, side $= e$, is revolved about one of its longer diagonals through 360°. Express the total surface and the volume in terms of e.

Ex. 7. Prove: The volume of a right triangular prism is equal to one half the product of any lateral face and the altitude to that face.

Ex. 8. Prove: If the diagonals of a quadrangular prism pass through a common point, the prism is a parallelopiped.

Ex. 9. Show how to pass a plane cutting a cube so that the section is a regular hexagon.

Ex. 10. The volume of a regular prism is equal to the product of its lateral area and one half the apothem of the base.

Ex. 11. What is the locus of all points equidistant from a plane and a line perpendicular to the plane?

Ex. 12. What is the ratio of the volumes of two cylinders of revolution that have equal lateral areas?

Ex. 13. What is the ratio of the lateral areas of two cylinders of revolution that have equal volumes?

BOOK VIII

THE SPHERE

702. Def. A **sphere** is a surface, all the points of which are equally distant from a fixed point. The fixed point is called the **center of the sphere**.

703. Def. A **radius** of a sphere is a straight line drawn from the center to any point on the sphere.

704. Def. A **diameter** of a sphere is a straight line passing through the center and limited by the surface.

705. From the definitions it follows that

(1) *All radii of a sphere* are equal, and all diameters are equal.

(2) A semicircle rotating about its diameter through an angle of 360° generates a sphere.

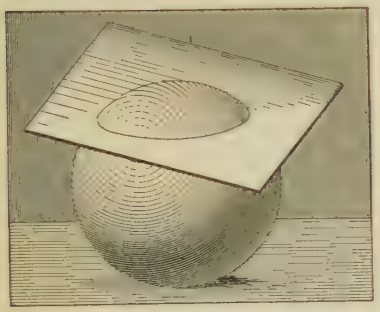
(3) *Two spheres are equal if their radii or their diameters are equal, and conversely.*

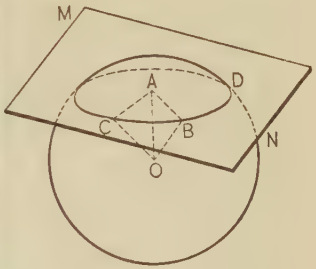
(4) *A point is without a sphere if its distance from the center is greater than a radius.*

Ex. The radii of two spheres are respectively 10 in. and 4 in., and the distance between their centers is 7 in. Does every point of the smaller sphere lie within the larger one?

PROPOSITION I. THEOREM

706. *Every section of a sphere made by a plane is a circle.*





Given CBD, the intersection of plane MN, and a sphere whose center is O.

To prove CBD is a circle.

Proof. From O draw $OA \perp MN$.

Draw AB, AC, CO, and OB. AC is a fixed line.

Since $OC = OB$,

$CA = AB$. (516)

Since AC is a fixed line, its equal AB must have the variable pt. B at a fixed distance from A.

Or CBD is a circle. (37)

707. COR. 1. *Circles of a sphere made by planes equidistant from the center are equal;* and of two circles made by planes not equidistant from the center, the circle made by the plane nearer to the center is the greater.

For since $\overline{AC}^2 = \overline{OC}^2 - \overline{AO}^2$, AC decreases as AO increases.

708. DEF. A **great circle of a sphere** is a section made by a plane passing through the center.

709. DEF. A **small circle of a sphere** is a section made by a plane not passing through the center.

710. DEF. The **axis of a circle of a sphere** is the diameter perpendicular to the plane of the circle; its ends are the **poles** of the circle.

711. COR. 2. *The axis of a circle passes through the center of the circle.*

712. COR. 3. *All great circles of a sphere are equal.*

713. COR. 4. *Any two great circles of a sphere bisect each other.*

For since the plane of each contains the center of the sphere, their intersection is a diameter and bisects both circles.

714. COR. 5. *Every great circle bisects the sphere.*

715. COR. 6. *One and only one circle may be drawn through any three points in the surface of a sphere.* (A plane is determined by three points.)

716. COR. 7. *A great circle may be drawn through two points B and C in the surface of a sphere.* (Three points, *B*, *C*, and the center *O*, determine a plane.)

Generally there is only one great circle which passes through two given points, but if the given points are ends of a diameter, any number of great circles can be passed through these points.

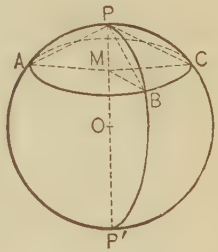
717. DEF. The **distance between two points on the surface of a sphere** is the length of the minor arc of the **great circle** between them.

EX. 1. Considering the earth as a sphere, what kind of circles are the meridians? The parallels of latitude? The equator? What are the poles of the parallels of latitude?

EX. 2. What is the radius of a small circle, if the distance of its plane from the center of the sphere is 9 in., and the radius of the sphere is 15 in.?

Proposition II. Theorem

718. *All points in a circle of a sphere are equidistant from a pole of the circle.*



Given P and P', the poles of circle ABC of a sphere.

To prove arc PA = arc PB = arc PC,

and arc $P'A$ = arc $P'B$ = arc $P'C$.

Hint. Prove by means of (711, 515) the equality of straight lines PA, PB, and PC.

719. Def. The **polar distance of a circle of a sphere** is the distance of a point in the circle from the nearer pole.

720. Remark. A **quadrant** in Spherical Geometry is the fourth part of a great circle.

721. Cor. 1. *The polar distance of a great circle is a quadrant.*

Cor. 2. *All the arcs of great circles drawn from a pole of a circle to its points are equal*

Ex. 1. The polar distance of a circle of a sphere is 60°, and the radius of the sphere is 13 in. Find

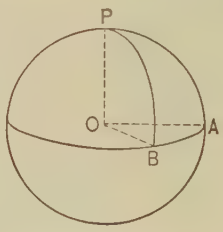
(*a*) the distance of its plane from the center.

(*b*) the radius of the circle.

Ex. 2. Considering the earth as a perfect sphere whose radius equals 4000 mi., find the polar distance of the Tropic of Cancer ($23\frac{1}{2}°$ N.) from the north pole.

Proposition III. Theorem

722. *On a sphere, a point at a quadrant's distance from two other points, not the extremities of a diameter, is the pole of the great circle passing through these two points.*



Given P, A, and B, three points on a sphere, and the quadrants, PA and PB.

To prove P is the pole of great circle AB.

Hint. Prove that the line OP is perpendicular to plane ABO by means of (502).

723. Remark. Theorem III enables us to construct a great circle through two points on the surface of a material sphere by means of compasses, provided we know the radius of the sphere.

From the given points, A and B, as centers draw arcs with a radius equal to the chord of a quadrant, intersecting in P. From P, with the same radius, draw a circle which is the required one.

724. Def. **A plane is tangent to a sphere** if it has one and only one point in common with the sphere.

725. DEF. **A line is tangent to a sphere** if, however far extended, it has one and only one point in common with the sphere.

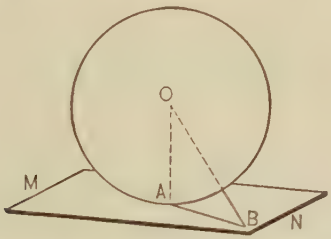
726. DEF. **Two spheres are tangent to each other** if they have one and only one point in common.

727. DEF. **A sphere is inscribed in a polyhedron** if every face of the polyhedron is tangent to the sphere.

728. DEF. **A sphere is circumscribed about a polyhedron** if every vertex of the polyhedron lies on the sphere.

PROPOSITION IV. THEOREM

729. *A plane perpendicular to a radius of a sphere at its outer extremity is tangent to the sphere.*



Given plane $MN \perp$ to radius OA of sphere O at A.

To prove MN is tangent to the sphere.

Proof. Join any point B in MN to O.

Then $OB > AO$. (518)

Hence B lies without the sphere. (705, (4))

Therefore, every point of MN, except A, lies without the sphere, and MN is tangent to the sphere.

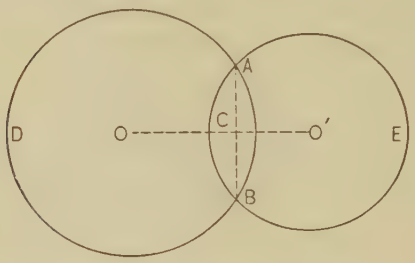
730. COR. *A plane (or line) tangent to a sphere is perpendicular to the radius drawn to the point of contact.* [Prove by indirect method.]

Ex. 1. Any straight line perpendicular to a radius at its outer extremity is tangent to the sphere.

Ex. 2. A straight line tangent to a circle of a sphere is a tangent to the sphere and lies in a plane that is tangent to the sphere at the point of contact.

Proposition V. Theorem

731. *The intersection of two spheres is a circle.*



Given ABD and ABE, two intersecting spheres whose centers are O and O'.

To prove the intersection of ABD and ABE is a circle.

Proof. The two intersecting spheres may be generated by the revolution of two intersecting circles ABD and ABE about their line of centers OO'. The line of intersection is then generated by the rotation of point A.

But OO' is the perpendicular-bisector of AB. (79)

$\therefore$ the line AB rotating generates a plane surface. (504)

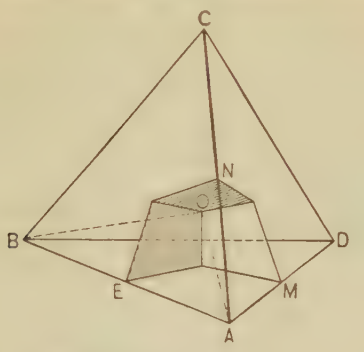
The point A (or B) rotating generates a circle whose radius is CA.

Hence the intersection of the two spheres is a circle.

732. Cor. *The plane of the circle of intersection of two spheres is perpendicular to the line of centers, and the center of the circle lies in the line of centers.*

PROPOSITION VI. THEOREM

733. *A sphere can be circumscribed about any tetrahedron.*



Given *ABCD*, a tetrahedron.

To prove a sphere may be circumscribed about *ABCD*.

Proof. At *E*, the mid-point of *AB*, construct the plane perpendicular to *AB*.

Similarly, construct planes perpendicular to *AC* and *AD* at their respective mid-points, *N* and *M*.

Let the three planes meet at *O*. (Why?)

$\therefore OA = OB.$ (507)

In like manner, $OA = OC$, and $OA = OD$.

Hence *O* is equidistant from *A*, *B*, *C*, and *D*, and a sphere described from *O* as a center with a radius equal to *AO* will be circumscribed about the tetrahedron.

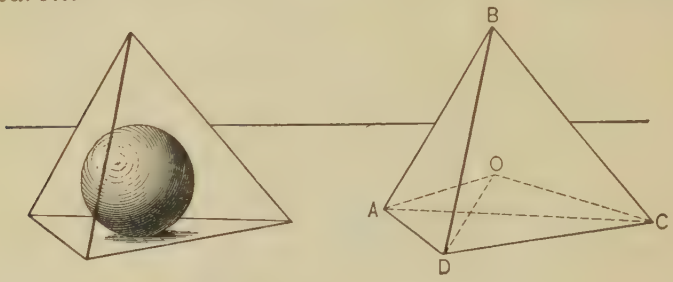
734. COR. *Four points not in the same plane determine a sphere.*

The six planes which are perpendicular to the six edges at their respective mid-points meet in a point.

Ex. There cannot be two circumspheres to a tetrahedron.

Proposition VII. Theorem

735. *A sphere can be inscribed in any given tetrahedron.*



Given *ABCD*, a tetrahedron.

To prove a sphere may be inscribed in *ABCD*.

Proof. Bisect any three dihedral angles as *AD*, *DC*, and *AC* by the planes *OAD*, *ODC*, and *OAC*.

Then *O* is equidistant from the four faces of the tetrahedron.

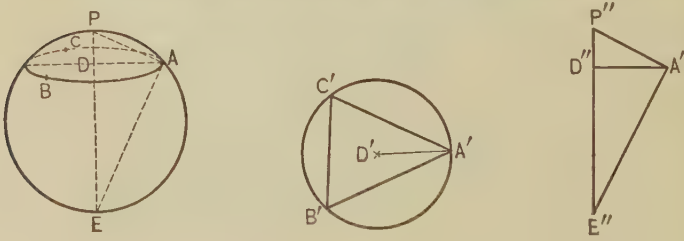
[To be completed by the student.]

736. Cor. *The six planes bisecting the dihedral angles of a tetrahedron meet in a point.*

Ex. The volume of a tetrahedron is equal to one third the product of its surface and the radius of the inscribed sphere.

Proposition VIII. Problem

737. *To construct the diameter of a material sphere.*



Given a material sphere $ABCE$.

Required the diameter of the sphere.

Construction. From any point P as center, with any [opening of compasses as] radius, draw circle ABC.

Measure [with compasses] the three chords AB, BC, and CA, and in any plane construct $\triangle A'B'C'$, having its sides respectively equal to AB, BC, and CA.

Construct $D'A'$ the radius of the circumscribed circle $A'B'C'$.

Draw right triangle $P''A''D''$, having the hypotenuse $P''A'' = PA$, and one arm $D''A'' = D'A'$.

At A'' draw $A''E'' \perp P''A''$, meeting $P''D''$ produced in E''.

Then $P''E''$ is the required diameter.

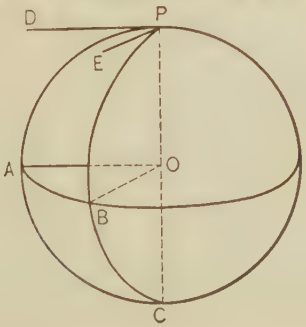
HINT. Prove $\triangle PAE \cong \triangle P''A''E''$.

738. DEF. The **angle between two intersecting curves** is the angle formed by the tangents at the point of intersection.

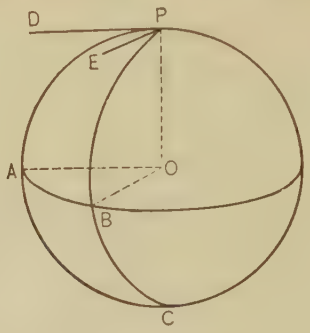
739. DEF. A **spherical angle** is the angle between two intersecting great circles.

PROPOSITION IX. THEOREM

740. *A spherical angle is measured by the arc of a great circle described from its vertex as a pole, and included between its sides, produced if necessary.*



Given DPE, the spherical angle formed by the great circles PAC and PBC; AB, an arc of a great circle having P for its pole.



To prove $\angle DPE$ is measured by $\overset{\frown}{AB}$.

Proof. Draw radii OA, OB, and OP.

$\angle POA$ is measured by quadrant PA. (227)

Hence $AO \perp OP$.

But $DP \perp OP$, (204)

and AO and DP are in the plane of $\odot PAO$.

Hence $AO \parallel DP$. (96)

Similarly $BO \parallel EP$.

$\therefore \angle DPE = \angle AOB$. (498)

But $\angle AOB$ is measured by $\overset{\frown}{AB}$. (227)

Hence $\angle DPE$ is measured by $\overset{\frown}{AB}$.

741. Cor. 1. *An angle formed by two great circles is equal to the plane angle of the dihedral angle formed by their planes.*

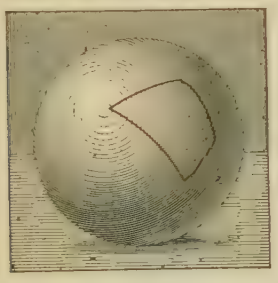
742. Cor. 2. *All great circles drawn through a pole of a great circle are perpendicular to it.*

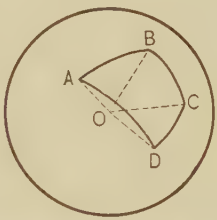
SPHERICAL POLYGONS

743. DEF. A **spherical polygon** is a figure on a sphere formed by three or more arcs of great circles.

The arcs are the **sides**, their points of intersection are the **vertices**, and the spherical angles formed by the sides are the **angles** of the polygon.

Thus, $ABCD$ is a spherical polygon, AB, BC, etc., its sides, A, B, C, etc., its vertices, and $\measuredangle ABC$, BCD, etc., its angles.





744. DEF. A **diagonal of a spherical polygon** is an arc joining any two nonadjacent vertices.

745. DEF. A **spherical triangle** is a spherical polygon of three sides. It is called **isosceles**, **equilateral**, etc., in the same cases in which a plane triangle would be so called.

The planes of the sides of a spherical polygon form at the center a polyhedral angle (O–$ABCD$) which is said to *correspond* with the spherical polygon.

The *sides* of the spherical polygon are measured by the *face angles* of the corresponding polyhedral angle; its *angles* are equal to the *dihedral angles* of the corresponding polyhedral angle.

746. REMARK. **By means of the relations between the parts of a spherical polygon and those of its corresponding polyhedral angle, we can deduce from any theorem of polyhedral angles an analogous theorem of spherical polygons.**

747. A spherical *polygon* is *convex* if its corresponding polyhedral angle is convex. All spherical polygons are supposed to be convex polygons unless stated otherwise.

748. Spherical *polygons* are *symmetric* if their corresponding polyhedral angles are symmetric. Evidently, their parts must be respectively equal, but follow in reverse order.

In general, two symmetric spherical polygons cannot be made to coincide.

The sides of a spherical polygon are usually measured in degrees.

Ex. State the theorems relating to spherical polygons which follow directly from the following propositions:

(*a*) Two trihedral angles are congruent or symmetric if their face angles are respectively equal.

(*b*) If two face angles of a trihedral angle are equal, the opposite dihedral angles are equal.

(*c*) If the three face angles of a trihedral angle are right angles, the three dihedral angles are right angles.

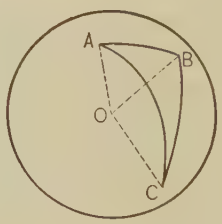
(*d*) The sum of the face angles of a polyhedral angle is less than 360°.

(*e*) The sum of two face angles of a trihedral angle is greater than the third face angle.

(*f*) If the opposite face angles of a tetrahedral angle are equal, the opposite dihedral angles are equal.

Proposition X. Theorem

749. *The sum of two sides of a spherical triangle is greater than the third side.*



Given ABC, a spherical triangle.

To prove $\overset{\frown}{AB} + \overset{\frown}{BC} > \overset{\frown}{AC}$.

Proof. Draw radii OA, OB, and OC.

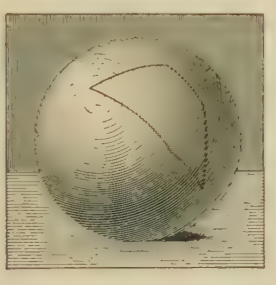
Then $\angle AOB + \angle BOC > \angle AOC.$ (557)

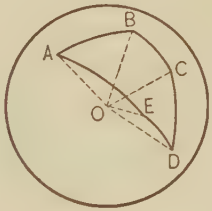
But the central angle is measured by the intercepted arc.

$\therefore \overset{\frown}{AB} + \overset{\frown}{BC} > \overset{\frown}{AC}.$

Proposition XI. Theorem

750. *The sum of the sides of a convex spherical polygon is less than the circumference of a great circle.*





Given $ABCDE$, a spherical polygon of n sides.

To prove $\overset{\frown}{AB} + \overset{\frown}{BC} + \overset{\frown}{CD} + \overset{\frown}{DE} + \overset{\frown}{EA}$, etc. $< 360°$.

Hint. Construct the corresponding polyhedral angle and compare Remark (746).

Proposition XII. Theorem

751. *Two triangles on the same sphere are congruent:*

(1) *If two angles and the included side of the one are respectively equal to two angles and the included side of the other,*

(2) *If two sides and the included angle of the one are respectively equal to two sides and the included angle of the other,*

(3) *If three sides of the one are respectively equal to three sides of the other,*

Provided the equal parts are arranged in the same order.

HINT. Prove the congruence of the corresponding polyhedral angles.

752. COR. *Two symmetric isosceles triangles are congruent.*

PROPOSITION XIII. THEOREM

753. *Two triangles on the same sphere are symmetric:*

(1) *If two angles and the included side of the one are respectively equal to two angles and the included side of the other,*

(2) *If two sides and the included angle of the one are respectively equal to two sides and the included angle of the other,*

(3) *If three sides of the one are respectively equal to three sides of the other,*

Provided the equal parts are arranged in the reverse order.

HINT. Prove that the corresponding polyhedral angles are symmetric.

754. REMARK. The equality of spherical angles and arcs is usually proved by means of equal or symmetric triangles.

PROPOSITION XIV. THEOREM

755. *The base angles of an isosceles spherical triangle are equal.*

HINT. Bisect the vertex angle and prove that two symmetric triangles are formed.

756. COR. *An equilateral spherical triangle is also equiangular.*

757. REMARK. Many theorems of Spherical Geometry may be proved by methods analogous to those of Plane Geometry.

758. NOTE. All figures to which the following exercises refer are supposed to be drawn on the surface of a sphere, and all constructions should be carried out on the surface of a spherical blackboard by means of a pair of compasses and a device for drawing arcs of great circles (*e.g.* the edge of a hemispherical cup).

In propositions relating to spherical figures, the words, *lines, radii, lines of centers, central angles,* etc., are often used for *arcs of great circles, polar distances, arcs joining the poles of two circles, angles formed at the pole,* etc.

EX. 1. Every point in the perpendicular bisector of an arc of a great circle is equidistant from the ends of the arc.

EX. 2. Two points equidistant from the ends of an arc of a great circle determine the perpendicular bisector of the arc.

Ex. 3. To bisect a spherical angle.

Ex. 4. To bisect an arc of a great circle.

Ex. 5. At a point in a given arc of a great circle, to draw a perpendicular to the arc.

Ex. 6. From a point without, to draw a perpendicular to a given arc of a great circle.

Ex. 7. If the opposite sides of a spherical quadrilateral are equal, the opposite angles are equal.

Ex. 8. Vertical spherical angles are equal.

Ex. 9. If the opposite sides of a spherical quadrilateral are equal, the diagonals bisect each other.

Ex. 10. To circumscribe a circle about a spherical triangle.

Ex. 11. At a given point in a great circle, to draw an angle equal to a given angle.

Ex. 12. To construct a spherical triangle having given two sides and the included angle.

Ex. 13. To construct a spherical triangle having given three sides.

Ex. 14. To construct a spherical triangle having given the base, the altitude, and the median corresponding with the base.

Ex. 15. The bisectors of the base angles of an isosceles spherical triangle are equal.

Ex. 16. The diagonals of an equilateral spherical quadrilateral are perpendicular to each other.

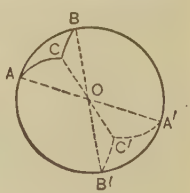
Ex. 17. A central angle of a circle on a sphere is measured by the intercepted arc.

Ex. 18. If two small circles on a sphere intersect, their line of centers bisects the common chord at right angles.

Ex. 19. If a radius bisects a chord of a small circle on a sphere, it is perpendicular to the chord.

Ex. 20. In a circle on a sphere, equal chords are equidistant from the pole.

759. DEF. Two spherical *polygons* are **vertical** if their corresponding polyhedral angles are vertical. (560)

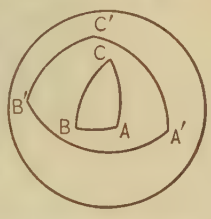


Ex. Two vertical spherical triangles are symmetric.

POLAR TRIANGLES

760. Def. If from the vertices of any spherical triangle as poles, arcs of great circles are described, another triangle will be formed which is called the **polar triangle** of the first.

Thus, if A is the pole of great circle $B'C'$, B the pole of great circle $A'C'$, and C the pole of great circle $A'B'$, $\triangle A'B'C'$ is the polar triangle of ABC.

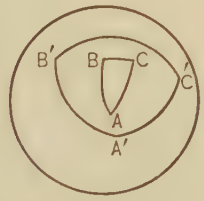


If from the poles A, B, and C entire circles should be described, eight triangles would be formed.

The polar triangle is selected by the following method: Denote the vertex formed by the intersection of the arcs from B and C by A', then of the two points of intersection that one is A' whose distance from A is less than a quadrant. Similarly for B' and C'.

Proposition XV. Theorem

761. *If one spherical triangle is the polar triangle of another, then the second is the polar triangle of the first.*

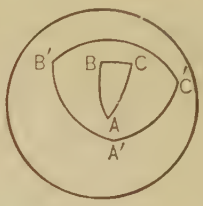


Given $\triangle A'B'C'$, the polar triangle of $\triangle ABC$.

To prove $\triangle ABC$ is the polar triangle of $\triangle A'B'C'$.

Proof. Since A is the pole of $B'C'$, the distance AB' is a quadrant, and since C is a pole of $A'B'$, $B'C$ is a quadrant.

Therefore, B' is the pole of AC. (722)



Similarly, A' is the pole of BC,

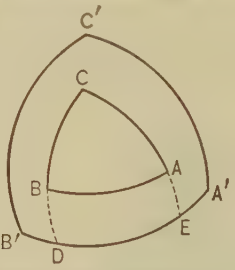
and C' is the pole of AB.

But the distances AA', BB', CC', are less than a quadrant.

Hence ABC is the polar triangle of $A'B'C'$.

PROPOSITION XVI. THEOREM

762. *In two polar triangles, each angle of one is measured by the supplement of the opposite side of the other.*



Given ABC and $A'B'C'$, polar triangles.

To prove $\angle C$ is the supplement of $B'A'$.

Proof. Produce the sides of $\angle C$ till they meet $B'A'$ in D and E respectively.

Then $\angle C$ is measured by arc DE, (740)

and $B'E$ and DA' are each 90°. (760)

But $DE + B'A' = B'E + DA'$ or 180°.

Hence DE and $B'A'$ are supplementary.

$\therefore$ $\angle C$ and $B'A'$ are supplementary.

Ex. 1. The sides of a spherical triangle are 60°, 50°, and 100°. Find the angles of the polar triangle.

763. Remark. **Polar spherical triangles are the means of proving propositions relating to angles by the corresponding propositions relating to sides, and *vice versa*.**

Ex. 2. What theorems relating to $\triangle ABC$ follow directly from the following propositions that relate to the polar triangle $A'B'C'$?

(*a*) If $A'B' = A'C'$, then $\angle B' = \angle C'$.

(*b*) $A'B' + B'C' + C'A' < 360°$.

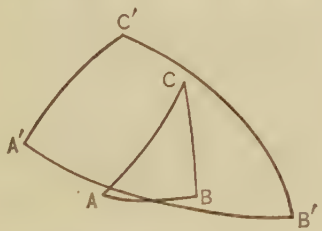
(*c*) $A'B' + B'C' > A'C'$.

Ex. 3. If in spherical $\triangle ABC$, $\angle A = 110°$, and $\angle B = 80°$, prove that $\angle C > 10°$.

Hint. Construct the polar $\triangle A'B'C'$.

Proposition XVII. Theorem

764. *The sum of the angles of a spherical triangle is greater than two and less than six right angles.*



Given ABC, a spherical triangle.

To prove $\angle A + \angle B + \angle C > 180°$,

and $\angle A + \angle B + \angle C < 540°$.

Proof. Construct the polar triangle $A'B'C'$, and denote the number of degrees in $B'C'$, $C'A'$, and $A'B'$, respectively, by a, b, and c.

Then
$$A = 180° - a. \tag{762}$$
$$B = 180° - b.$$
$$C = 180° - c.$$

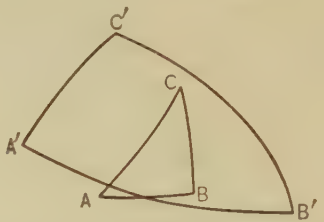
Adding the equations, we obtain

$$A + B + C = 540° - (a + b + c).$$

Hence $A + B + C < 540°.$

But $a + b + c < 360°.$ (740)

Hence $A + B + C > 180°.$

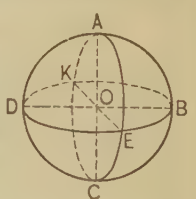


765. Cor. A spherical triangle may have one, two, or three right angles; also one, two, or three obtuse angles.

766. Def. A **bi-rectangular spherical triangle** is a spherical triangle that has two right angles.

767. Def. A **tri-rectangular spherical triangle** is a spherical triangle that has three right angles.

768. Cor. 1. *If three planes be passed through the center of a sphere, each perpendicular to the other two, the surface is divided into eight equal tri-rectangular triangles having all their sides equal to quadrants.*



769. Cor. 2. *The surface of a sphere is equal to eight times the surface of a tri-rectangular triangle.*

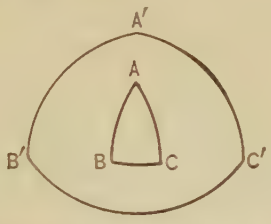
Ex. 1. An exterior angle of a spherical triangle is less than the sum of the two remote interior angles.

Ex. 2. The sum of the angles of a spherical quadrilateral is greater than four right angles.

Ex. 3. A tri-rectangular triangle coincides with its polar triangle.

Proposition XVIII. Theorem

770. *If two angles of a spherical triangle are equal, the sides opposite are equal.*



Given spherical triangle ABC, with $\angle B = \angle C$.

To prove $\overset{\frown}{AB} = \overset{\frown}{AC}$.

Proof. Construct polar $\triangle A'B'C'$.

Then $\overset{\frown}{A'B'}$ is the supplement of $\angle C$, and $\overset{\frown}{A'C'}$ the supplement of $\angle B$. (762)

Hence $\overset{\frown}{A'B'} = \overset{\frown}{A'C'}$.

$\therefore \angle B' = \angle C'$. (755)

But $\overset{\frown}{AB}$ and $\overset{\frown}{AC}$ are respectively the supplements of $\angle C'$ and $\angle B'$. (762)

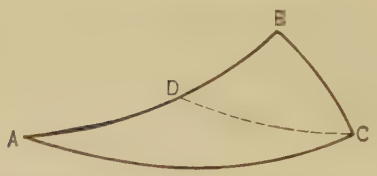
Whence $\overset{\frown}{AB} = \overset{\frown}{AC}$.

Ex. **1.** An equiangular spherical triangle is also equilateral.

Ex. **2.** If on the three sides AB, BC, and CA of the equilateral spherical triangle ABC, equal distances AA', BB', and CC' be laid off, then $A'B'C'$ is also an equilateral triangle.

Proposition XIX. Theorem

771. *In any spherical triangle, the greater side is opposite the greater angle.*



Given spherical triangle ABC, with $\angle BCA > \angle BAC$.

To prove $\overset{\frown}{AB} > \overset{\frown}{BC}$.

Proof. Draw an arc of a great circle CD so that $\angle DCA = \angle A$, and let D be the intersection of $\overset{\frown}{AB}$ and $\overset{\frown}{DC}$.

Then	$\overset{\frown}{AD} = \overset{\frown}{DC}$.	(770)
But	$\overset{\frown}{DB} + \overset{\frown}{DC} > \overset{\frown}{BC}$.	(749)
	$\therefore \overset{\frown}{DB} + \overset{\frown}{AD} > \overset{\frown}{BC}$.	(Sub.)
Or	$\overset{\frown}{AB} > \overset{\frown}{BC}$.	

Ex. If $\overset{\frown}{BC}$ is the base of an isosceles spherical triangle ABC, and D any point in $\overset{\frown}{AC}$, then $\overset{\frown}{BD} > \overset{\frown}{CD}$.

Proposition XX. Theorem

772. *In any spherical triangle, the greater angle is opposite the greater side.*

Given in spherical $\triangle ABC$, $\overset{\frown}{AB} > \overset{\frown}{BC}$.

To prove $\angle C > \angle A$.

Hint. Prove by the indirect method.

Ex. Prove Prop. XX by means of polar triangles.

Proposition XXI. Theorem

773. *If two triangles on the same sphere are mutually equiangular, they are mutually equilateral, and either congruent or symmetrical.*

Given spherical $\triangle A$ and A' mutually equiangular.

To prove A and A' mutually equilateral and either equal or symmetrical.

Proof. Construct P the polar triangle of A, and P' the polar triangle of A'.

Since A and A' are mutually equiangular, (Hyp.)

P and P' are mutually equilateral. (762)

Hence P and P' are mutually equiangular. (751, 753)

But A and A' are the polar triangles of P and P'. (761)

Hence A and A' are mutually equilateral. (762)

Hence A and A' are either equal or symmetrical.

Ex. 1. If the opposite angles of a spherical quadrilateral are equal, the opposite sides are equal.

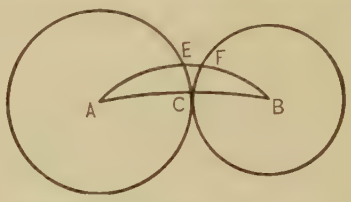
Hint. Produce two opposite sides in both directions until they meet, and find two congruent triangles.

Ex. 2. If in spherical quadrilateral $ABCD$, $\angle A = \angle B$, and $\angle C = \angle D$, then $BC = AD$. (Hint. Ex. 1.)

Ex. 3. If two points A and B whose distance is 180° are joined by two semicircles, ACB and ADB, and $\angle BCD = \angle CDA$, then $BC = AD$.

Proposition XXII. Theorem

774. *The shortest line between two points on the surface of a sphere is the minor arc of a great circle.*



Given AB, the minor arc of a great circle, on a given sphere.

To prove AB is shorter than any other line of the sphere joining A and B.

Proof. Take any point C on AB, and from A and B as centers, with radii respectively equal to AC and BC, draw two circles. These two circles cannot meet in any other point, for if they should meet in some point D, the sum of two sides AD and DB of $\triangle ADB$ would not be greater than the third side AB.

Any other line $AEFB$ between A and B must, therefore, intersect the circumferences in two points E and F, respectively.

But $AEFB$ cannot be the shortest line between A and B, for by revolving AE about A, and BF about B till E and F coincide with C, we would obtain a line joining A and B which is shorter than $AEFB$.

Hence, the shortest line between A and B must pass through C.

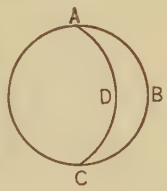
But since C is any point in AB, the shortest line between A and B must coincide with AB.

Or AB is the shortest line joining A and B.

Ex. New York and Oporto (Portugal) lie (approximately) on the same parallel of latitude. If a ship sailed from New York to Oporto by remaining always on the parallel of latitude, would its course be the shortest possible course?

MEASUREMENT OF SPHERICAL FIGURES

775. DEF. A **lune** is a figure on a sphere formed by two semicircles, as *ABCD*.



776. DEF. An **angle of the lune** is one of the spherical angles formed by the arcs.

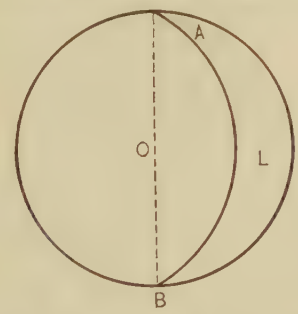
Evidently, lunes on the same sphere are equal if their angles are equal. Hence if the angle of a lune is divided into n equal parts, each of the small lunes thus formed is the nth part of the given lune.

777. In spherical geometry the area of a **tri-rectangular triangle** is often taken as a **unit** of surface, and the right angle as the angular unit.

The area of a sphere is then measured by 8, and the area of a hemisphere by 4.

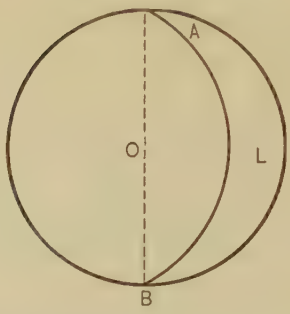
PROPOSITION XXIII. THEOREM

778. *If the right angle is taken as angular unit, and the tri-rectangular triangle as the unit of surface, the area of a lune is measured by twice one of its angles.*



Given L denoting the number of tri-rectangular triangles contained in a lune AB, and A the number of right angles in its angle.

To prove $L = 2A$.



Proof. We may consider the entire surface of the sphere as a lune whose angle = 4 rt. $\measuredangle$.

Or if $A = 4, L = 8.$

$\therefore$ If $A = 1, L = 2.$ (776)

$\therefore$ If $A = \frac{1}{n}, L = \frac{2}{n}.$ (776)

$\therefore$ If $A = \frac{m}{n}, L = \frac{2m}{n}.$

Or $L = 2A$, if A is a rational number.

If A is irrational, all successive approximations of $2A$ and L are respectively equal.

Hence $L = 2A.$ (223)

779. Cor. 1. *Two lunes on the same sphere, or equal spheres, are to each other as their angles.*

780. Cor. 2. *A lune is to the surface of the sphere on which it lies as its angle is to four right angles.*

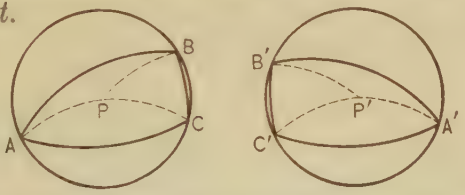
Ex. 1. Find the number of tri-rectangular triangles contained in a lune, if its angle equals (*a*) $1\frac{1}{2}$ rt. $\measuredangle$, (*b*) 45°.

Ex. 2. Let A denote the angle of a lune, Tr the area of a tri-rectangular triangle on a sphere whose surface equals S. Find the area of the lune if

(*a*) $A = 2\frac{1}{2}$ rt. $\measuredangle$, $Tr = 20$ sq. ft.
(*b*) $A = 60°$, $Tr = 30$ sq. ft.
(*c*) $A = \frac{1}{3}$ rt. $\angle$, $S = 80$ sq. in.
(*d*) $A = 20°$, $S = 4$ sq. in.

PROPOSITION XXIV. THEOREM

781. *Symmetric triangles on the same sphere are equivalent.*



Given ABC and $A'B'C'$, symmetric, spherical triangles on same sphere.

To prove $\triangle ABC = \triangle A'B'C'$.

Proof. Let P and P' be the poles of small circles passing through A, B, C and A', B', C', respectively.

Since arcs AB, BC, CA are equal respectively to arcs $A'B'$, $B'C'$, $C'A'$, chords AB, BC, CA are equal respectively to chords $A'B'$, $B'C'$, $C'A'$. (187)

Hence plane $\measuredangle ABC$ and $A'B'C'$ are congruent. (74)

Hence circle ABC = circle $A'B'C'$. (193)

Draw PA, PB, PC, $P'A'$, $P'B'$, and $P'C'$.

These arcs are equal. (718)

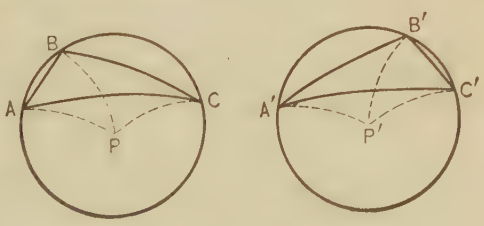
Hence $\triangle ABP \cong \triangle A'B'P'$. (752)

Similarly, $\triangle ACP \cong \triangle A'C'P'$,

$\triangle BCP \cong \triangle B'C'P'$.

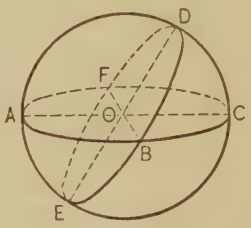
Hence by addition $\triangle ABC = \triangle A'B'C'$.

782. REMARK. If the poles of $\triangle ABC$ and of $\triangle A'B'C'$ lie without the triangles, the proof has to be slightly modified.



HINT. $\triangle ABC = \triangle ABP + \triangle BCP - \triangle ACP$.

783. Cor. *If two arcs of great circles ABC and DBE intersect on a hemisphere, the sum of the areas of the two opposite spherical triangles ABE and DBC is equivalent to a lune whose angle is equal to the angle ABE, included by the two arcs.*



Hint.

$\widehat{AB} = \widehat{CF}$, $\widehat{EB} = \widehat{DF}$, $\widehat{AE} = \widehat{DC}$.

$\therefore \triangle ABE$ is symmetric to $\triangle DCF$.

$\therefore \triangle ABE = \triangle DCF$, etc.

784. Def. The **spherical excess** of a spherical polygon is the excess of the sum of all its angles over the sum of the angles of a plane polygon of the same number of sides. In this book the spherical excess is *always measured in right angles.*

Thus, if A, B, and C are the three angles of a triangle expressed in right angles, and the spherical excess is denoted by E, then

$$E = A + B + C - 2.$$

If A, B, C, $\cdots$, are the angles of a polygon of n sides, expressed in right angles, then

$$E = (A + B + C + \cdots) - 2(n - 2).$$

Ex. 1. Find the spherical excess of a spherical triangle ABC, if

(*a*) $A = \frac{1}{2}$ rt. $\angle$, $B = \frac{3}{4}$ rt. $\angle$, $C = 1$ rt. $\angle$.
(*b*) $A = 60°$, $B = 70°$, $C = 80°$.
(*c*) $A = 90°$, $B + C = 180°$.

Ex. 2. Find the spherical excess of a spherical quadrilateral $ABCD$, if

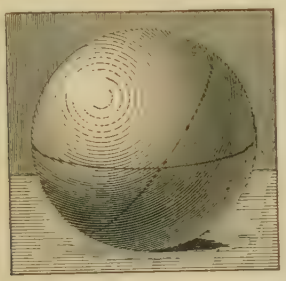
(*a*) $A = 90°$, $B = 100°$, $C = 110°$, $D = 80°$.
(*b*) $A = 150°$, $B = 110°$, $C + D = 190°$.

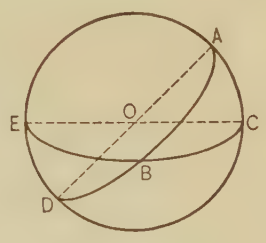
Ex. 3. Find the spherical excess of a spherical hexagon, if each of its angles equals

(*a*) 130°.
(*b*) 170°.

Proposition XXV. Theorem.

785. *If the right angle be taken as angular unit, and the tri-rectangular triangle as unit surface, the area of a spherical triangle is measured by the spherical excess.*





Given ABC, a spherical triangle.

To prove area $ABC = A + B + C - 2$, if the angular unit is the right angle, and the unit of surface the tri-rectangular triangle.

Proof. Complete the circle $ACDE$, and produce $\overset{\frown}{AB}$ and $\overset{\frown}{CB}$ until they meet ACD in D and E respectively.

Then, since $\triangle ABC + \triangle BED$ is equivalent to the lune, whose angle is equal to $\angle ABC$, (783)

$$\triangle ABC + \triangle BED = 2B. \quad (778)$$

But $$\triangle ABC + \triangle ABE = 2C, \quad (778)$$

and $$\triangle ABC + \triangle BCD = 2A. \quad (778)$$

By adding these equations, we obtain

$$3\triangle ABC + \triangle BED + \triangle ABE + \triangle BCD = 2(A + B + C).$$

But $\triangle ABC + \triangle BED + \triangle ABE + \triangle BCD =$ a hemisphere or 4.

Hence $$2\triangle ABC + 4 = 2(A + B + C),$$

or $$\triangle ABC + 2 = A + B + C.$$

Whence $$\triangle ABC = A + B + C - 2.$$

786. COR. *If E denotes the spherical excess of a spherical triangle lying on a sphere whose surface is S, the area of the triangle equals* $E \times \frac{S}{8}$.

Ex. 1. If the surface of a sphere is 40 sq. ft., find the area of a spherical triangle whose angles are respectively 40°, 60°, and 100°.

Ex. 2. What fraction of the surface of a sphere is covered by a spherical triangle whose angles are 90°, 100°, and 110°, respectively?

Ex. 3. How large is the angle of an equiangular spherical triangle whose area is one fourth of the surface of the sphere?

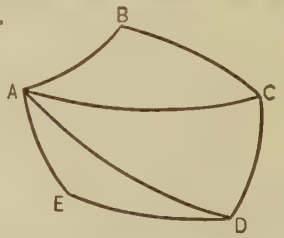
Ex. 4. What is the ratio of a lune whose angle is equal to 80° to the area of an equilateral spherical triangle whose angle is equal to 80°?

Ex. 5. The arms of an isosceles spherical triangle whose base angles are 80° are produced so as to form a lune. Find the vertex angle if the triangle is the third part of the lune.

Ex. 6. The angles of a spherical triangle are 80°, 90°, and 100°. Find the angle of an equivalent lune.

PROPOSITION XXVI. THEOREM

787. *If the right angle be taken as the angular unit, and the tri-rectangular triangle as unit of surface, the area of any spherical polygon is measured by the spherical excess.*



Given $ABCD \cdots$, a spherical polygon of n sides.

To prove $ABCD \cdots$ measured by $(A + B + C + \cdots) - 2(n-2)$.

Proof. Draw all diagonals from A. which will divide $ABCD \cdots$ into $(n-2)$ triangles.

The area of each triangle is equal to the sum of its angles less 2. (785)

Hence the area of the polygon $ABCD \cdots$ is equal to

$$(A + B + C + \cdots) - 2(n - 2).$$

788. Remark. Props. XXV and XXVI give us the relative, but not the absolute, size of areas on the sphere.

Ex. 1. Find the area of an equiangular hexagon whose angle is 160°. (The tri-rectangular triangle is taken as unit.)

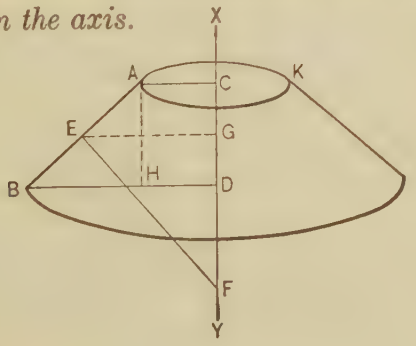
Ex. 2. What fraction of the surface of a sphere is covered by a hexagon whose angles are 130°, 140°, 130°, 150°, 120°, 160°?

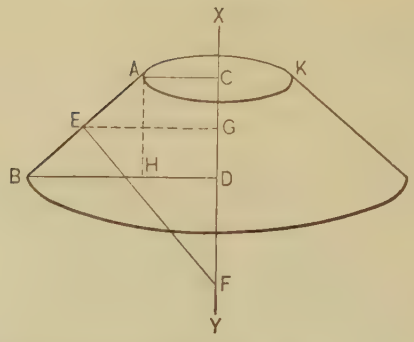
Ex. 3. The angles of a spherical quadrilateral are 90°, 100°, 110°, and 120°. Find the angle of an equivalent equilateral triangle.

Ex. 4. The distance between the centers of two spheres is 13. Find the radius of the circle of intersection if the radii of the spheres are 5 and 12.

Proposition XXVII. Theorem

789. *The area generated by the revolution of a straight line about an axis in its plane is equal to the projection of the line upon the axis multiplied by the circumference of a circle whose radius is the perpendicular erected at the mid-point of the line and terminating in the axis.*





Given AB revolving about an axis XY in its plane, generating surface ABK, CD the projection of AB upon XY, and EF the perpendicular bisector of AB terminating in XY.

To prove area $ABK = CD \times 2\pi EF$.

Proof. Draw $EG \parallel BD$ and $AH \parallel CF$.

The surface ABK is the lateral surface of a frustum of a cone of revolution.

Hence area $ABK = AB \times 2\pi EG$. (1) (698)

But $\triangle ABH \sim \triangle EFG$. (306)

$$\therefore \frac{AB}{EF} = \frac{AH}{EG}. \quad (303)$$

Whence $AB \times EG = AH \times EF$, (274)

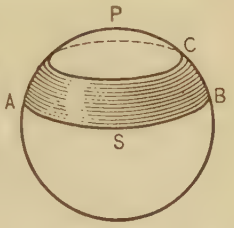
or $AB \times EG = CD \times EF$. (Sub.)

Substituting this value in (1),

$$\text{area } ABK = CD \times 2\pi EF.$$

Ex. A chord is drawn in a semicircle having a diameter of 26 ft. The chord is 12 ft. from the center of the semicircle and the center of the chord is 6 ft. from the diameter. What is the surface generated by revolving the chord about the diameter?

790. DEF. A **zone** is the portion of a sphere included between two parallel planes.



791. DEF. The circles common to the sphere and the planes form the **bases** of the *zone*, and the distance between the two planes is the **altitude** of the zone.

Thus the portion ABC of the sphere S represents a zone. If one of the two planes is tangent to the sphere, and the other intersects the sphere, they intercept a *zone of one base*, as PAB.

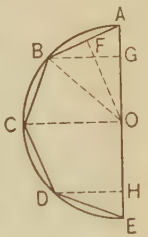
Ex. 1. Prove Prop. XXVII, when $AB \parallel XY$.

Ex. 2. Prove Prop. XXVII, when A lies in XY.

Ex. 3. In the diagram for Prop. XXVII, find the area ABK, if EF is equal to 10 in., AB is equal to 8 in., and angle $ABH = 30°$.

PROPOSITION XXVIII. THEOREM

792. *The area of a sphere equals four times the area of a great circle.*



Given S, the area of the sphere, R, the radius of the sphere generated by revolving semicircle ABE about AE.

To prove $S = 4\pi R^2$.

Proof. Inscribe in the semicircle half of a regular polygon of an even number of sides, as $ABCDE$.

Draw BG, CO, and DH perpendicular to AE, and denote the common distance of the chords AB, BC, CD, and DE from O by d.

Then area AB [1] $= AG \times 2\pi d$, (791, Ex. 2)

area $BC = GO \times 2\pi d$, etc. (789)

[1] Area AB means area generated by AB.

Adding the equations

area $ABCDE = (AG + GO + OH + HE) \times 2\pi d$

or area $ABCDE = 2R \times 2\pi d = 4R\pi d$.

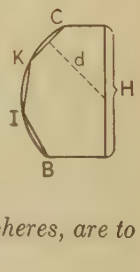
Now let the number of sides of the inscribed polygon be increased indefinitely. Then the area $ABCDE$ approaches S, and d approaches R as a limit.

Hence, $$S = 4\pi R^2. \qquad (414)$$

793. Cor. 1. *The areas of two spheres are to each other as the squares of their radii.*

794. Cor. 2. *The area of a zone is equal to its altitude multiplied by the circumference of a great circle.*

The zone generated by the revolution of arc BC is equal to $H \times 2\pi R$, for, if we divide $\overset{\frown}{BC}$ into equal parts and join the successive points of division, it follows in the same manner as in (792) that



$$\text{area } BIKC = H \cdot 2\pi d.$$

Hence, $$\text{zone} = H \cdot 2\pi R. \qquad (414)$$

795. Cor. 3. *Zones on the same, or on equal spheres, are to each other as their altitudes.*

796. Remark. Prop. XXVIII, in connection with Props. XXV and XXVI, enables us to find the absolute size of the spherical surfaces.

Ex. 1. Assuming $\pi = 3.1416$, find the areas of spheres whose radii are

(*a*) 10 in., (*b*) 4 ft., (*c*) 3 yd., (*d*) 2 ft. 4 in., (*e*) 4000 mi.

Ex. 2. Find the radii of spheres whose areas are

(*a*) 314.16 sq. ft., (*b*) 628.32 sq. in., (*c*) 1 sq. ft., (*d*) 10π sq. ft.

Ex. **3**. On a sphere whose radius equals 10 in., find the areas, in sq. in., of lunes whose angles are

(*a*) 30°, (*b*) 45°, (*c*) 90°, (*d*) 135°.

Ex. **4**. Compute the area of a zone whose altitude is h, lying on a sphere whose radius equals R, if

(*a*) $R = 20$ in., $h = 10$ in.

(*b*) $R = 4$ ft., $h = 3$ ft.

(*c*) $R = 1$ ft., $h = .001$ ft.

Ex. **5**. If R is the radius of the sphere, and $\pi = \frac{22}{7}$, find the areas of spherical triangles whose angles are A, B, and C, if

(*a*) $A = 60°$, $B = 70°$, $C = 85°$, $R = 10$ ft.

(*b*) $A = 70°$, $B = 80°$, $C = 72°$, $R = 1$ mi.

(*c*) $A = 90°$, $B = 90°$, $C = 90°$, $R = 1$ in.

(*d*) $A = 80°$, $B = 90°$, $C = 100°$, $R = 20$ ft.

Ex. **6**. If R is the radius of the sphere, find the areas of spherical polygons whose angles are A, B, C, D, etc., if

(*a*) $A = 90°$, $B = 100°$, $C = 110°$, $D = 130°$, $R = 4$ ft.

(*b*) $A = B = C = D = E = F = 134°$, $R = 10$ in.

(*c*) $A = 70°$, $B = 90°$, $C = 100°$, $D = 120°$, $R = 4$ yd.

Ex. **7**. Find the number of square feet in a hemispherical dome whose diameter is 40 ft.

Ex. **8**. If we assume the earth to be a perfect sphere having a radius of 4000 mi., what is the area of its surface?

Ex. **9**. If the radius of the earth is 4000 mi. and the altitude of the north temperate zone is $\frac{13}{28}$ of the radius, what is the area of the north temperate zone?

Ex. **10**. Find the area of the zone bounded by the equator and the parallel 30° N., if the radius of the earth equals 4000 mi.

Ex. **11**. Divide a sphere into 10 equivalent zones.

Ex. **12**. The surface of a sphere equals 110 sq. in., and a zone lying on the sphere is equivalent to 11 sq. in. Find the altitude of the zone.

Ex. **13**. Find the angle of an equilateral spherical triangle on a sphere, whose radius is 2, if the area of the triangle is equal to π.

SPHERICAL VOLUMES

797. DEF. A **spherical sector** is the figure generated by any plane sector of a semicircle when the semicircle revolves about its diameter.

The **base** of the spherical sector is the zone generated by the revolution of the arc.

Thus, if the sector OAB, Fig. 2, revolves about the diameter XY, it generates the spherical sector O–AB (Fig. 1).

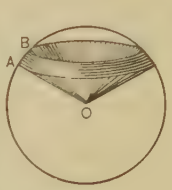


FIG. 1

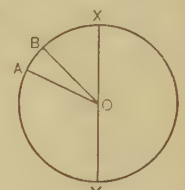


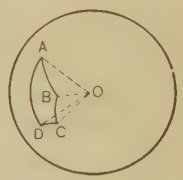
FIG. 2

If one radius of the plane sector that generates a spherical sector forms a part of the axis, the base of the spherical sector is a zone of one base, and the spherical sector is sometimes called a *spherical cone.*

798. DEF. A **spherical pyramid** is the figure bounded by a spherical polygon and the faces of the corresponding polyhedral angle.

The spherical polygon is called the **base** and the center of the sphere the **vertex** of the spherical pyramid.

Thus, O–$ABCD$ is a spherical pyramid having base $ABCD$.



799. DEF. A **spherical segment** is the figure formed by two *parallel* planes and the part of the sphere between the planes.

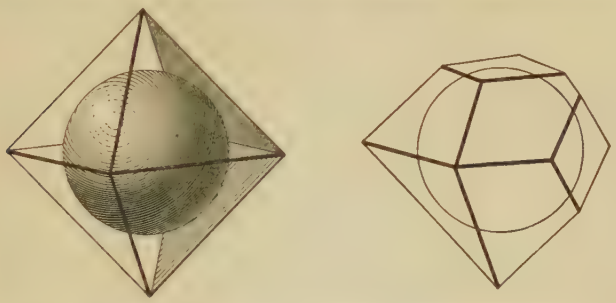
The sections made by the parallel planes are the **bases,** and the distance between the planes is the **altitude** of the segment.

If one of the bounding planes is tangent to the sphere, the segment is called a **segment of one base.**

800. DEF. A **spherical wedge** is the figure formed by a lune and the planes of its sides.

PROPOSITION XXIX. THEOREM

801. *The volume of a sphere is equal to one third of its radius multiplied by the area of the sphere.*



Given V, the volume of a sphere whose radius is R and whose surface is S.

To prove $V = S \times \frac{1}{3} R$.

Proof. Conceive any polyhedron circumscribed about the sphere, and let each vertex of the polyhedron be joined to the center of the sphere.

The polyhedron is then divided into pyramids, each having one face as a base and R as the common altitude.

Therefore, the volume of the polyhedron is equal to the sum of the bases multiplied by one third of R, or it is equal to the surface multiplied by one third of the radius.

But by increasing the number of faces of the polyhedron indefinitely, its volume will approach the volume of the sphere and its area, the area of the sphere as a limit.

Hence $V = S \times \frac{1}{3} R.$ (414)

802. COR. 1. *If V denotes the volume, R the radius, and D the diameter of the sphere,*

$$V = \tfrac{4}{3} \pi R^3 \text{ or } V = \frac{\pi}{6} \times D^3 = \frac{\pi D^3}{6}.$$

803. Cor. 2. *The volumes of two spheres are to each other as the cubes of their radii, or of their diameters.*

804. Cor. 3. *The volume of a spherical pyramid is equal to the area of its base multiplied by one third the radius of the sphere.*

805. Cor. 4. *The volume of a spherical sector is equal to the area of its base multiplied by one third the radius of the sphere.*

The proof is analogous to the proof of Prop. XXIX.

806. Cor. 5. *If H denotes the altitude of a spherical sector, V its volume, and Z the surface of the base,*

$$V = Z \times \frac{R}{3}.$$

But
$$Z = 2\pi R \times H. \tag{794}$$

$$\therefore V = \tfrac{2}{3}\pi R^2 H.$$

807. Cor. 6. *The volume of a sphere is to the volume of the circumscribed cylinder as 2 to 3.*

For the volume of the circumscribed cylinder is $2\pi R^3$.

Ex. 1. Find the volumes of spheres whose radii are

(*a*) 10 in., (*b*) 2 in., (*c*) 4 in., (*d*) 1 ft., (*e*) 4000 mi.

Ex. 2. Assuming $\pi = \frac{22}{7}$, find the radii of spheres whose volumes are

(*a*) 4851 cu. ft., (*b*) .1 cu. ft.

Ex. 3. Find the volume of a spherical sector whose base is 20, if the radius of the sphere is 10.

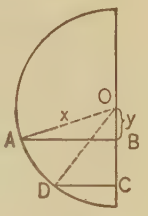
Ex. 4. Find the radius of a sphere whose volume is 100 cu. ft.

Ex. 5. Find the volume of a sphere whose area is equal to 10 sq. m.

Ex. 6. Find the area of a sphere whose volume is equal to M

Note. The volume generated by an area $(ABC\cdots)$ rotating about an axis XY will be designated by $V(ABC\cdots)$.

Ex. 7. To find the volume of a spherical segment generated by $ABCD$, if $AB = 6$ in., $BC = 2$ in., $DC = 4$ in., and $\angle B = \angle C = 90°$.



Solution. Draw OA, and OD. Denote OA by x and OB by y.

Since $\angle B = 90°$, $x^2 = y^2 + 36$.

Since $\angle C = 90°$, $x^2 = (y + 2)^2 + 16$.

Subtracting, $0 = y^2 - (y + 2)^2 + 20$.

$\therefore 4y = 16$.

$y = 4$.

Hence, $x^2 = 16 + 36$.

Or $x = \sqrt{52} = R$.

But segment $= V(ABCD)$,

and $V(ABCD) = V(OAD) + V(ODC) - V(OAB)$.

$$V(OAD) = \text{zone} \times \frac{R}{3} = 2\pi R \times H \times \frac{R}{3} = 2\pi \cdot 2 \cdot \frac{52}{3} = \frac{208\pi}{3}.$$

$$V(ODC) = (CD)^2\pi \cdot \frac{OC}{3} = \frac{16 \cdot \pi \cdot 6}{3} = \frac{96\pi}{3}.$$

$$V(OAB) = (AB)^2\pi \cdot \frac{OB}{3} = \frac{36\pi \cdot 4}{3} = \frac{144\pi}{3}.$$

$$\therefore V(ABCD) = \frac{\pi}{3}(208 + 96 - 144) = \frac{160\pi}{3}.$$

Ex. 8. Find the volume of a spherical segment, the radii of whose bases are 2 and 5, and whose altitude is 1.

Ex. 9. Find the volume of a spherical segment, the radii of whose bases are 6 and 8, and whose altitude is 2.

Ex. 10. Find the volume of a spherical segment of one base the radius of whose base is 3, and whose altitude is 2.

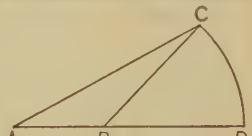
Ex. 11. Find the volume generated by the rotation of triangle ABC about AB as an axis if $AB = 14$, $BC = 15$, and $CA = 13$.

Ex. 12. Find the volume generated by the rotation of trapezoid $ABCD$ about its base AB, if $AB = 10$, $BC = AD = 5$, and $CD = 4$.

Ex. 13. A line XY and a triangle ABC lie in one plane, A lies in XY, and from C, CC' is drawn perpendicular to XY, and similarly $BB' \perp XY$. Find the volume generated by the rotation of triangle ABC about XY, if $AC' = 5$, $C'B' = 7$, $AB' = 12$, $CC' = 12$, and $BB' = 5$.

Ex. **14**. In the figure, ABD is a straight line and B is the center of arc CD. Find the volume generated by the rotation of the entire figure about AD as an axis, if $AC = 15$, $CB = 13$, and $AB = 4$.

Ex. **15**. In the same diagram find the volume generated by the rotation of triangle ABC about AB as axis, if $AB = 5$, $BC = 8$, and $\angle ABC = 120°$.



Exs. 14, 15

Ex. **16**. The volumes of two spheres are to each other as 8 to 125. Find the ratio of their radii.

Ex. **17**. The volumes of two spheres are to each other as 125 to 216. Find the ratio of their areas.

Ex. **18**. Find the radius of a sphere whose area is equivalent to the sum of the areas of two spheres whose radii are 3 and 4 respectively.

Ex. **19**. Find the volume of a spherical shell whose exterior radius is 13 and whose thickness is 8.

Ex. **20**. Find the radius of a sphere equivalent to the spherical shell whose exterior radius is 3 and whose thickness is 1.

Ex. **21**. If the diameter of the moon is 2160 mi., find its area and its volume.

Ex. **22**. The area of a zone on a sphere is 80, its altitude 4. Find the radius of the sphere.

Ex. **23**. Find the radius of a sphere equivalent to a cube whose edge is equal to a.

Ex. **24**. A cylindrical vessel 4 in. in diameter, is partly filled with water. Upon immersing a ball in the water, the surface rises 1 in. Find the diameter of the ball.

Ex. **25**. A sphere whose radius is 2 in. weighs 32 oz. Find the weight of a sphere of the same material whose radius is 3 in.

Ex. **26**. Find the volume of a spherical pyramid whose base is an equilateral triangle with each angle equal to 80°, if the radius of the sphere is equal to 10.

Ex. **27**. A square whose side is 4 revolves about one of its diagonals. Find the area and the volume of the generated solid.

Ex. **28**. Find the volume of a spherical segment of one base, if the area of its curved surface is 20π and its altitude is 2.

Ex. **29**. Find the radius of a sphere whose area is equal to the total area of a cube whose edge is equal to 4.

Ex. **30**. The edge of a cube is 10 in. Find the diameter of the circumscribed sphere.

Ex. **31**. Find the radius of a sphere inscribed in a regular tetrahedron whose edge equals 4 in.

Ex. **32**. A lune whose angle is equal to 40° is equal to a zone on the same sphere. Find the altitude of the zone if the diameter of the sphere is 18 in.

Ex. **33**. The dihedral angles of a spherical pyramid of six sides are each 140°. Find the volume of the pyramid if the radius is equal to 10.

Ex. **34**. The area of a sphere is equal to the lateral area of the circumscribed cylinder.

Ex. **35**. Find the ratio of the volume of a sphere to that of its circumscribed cube.

Ex. **36**. If the diagonals of a spherical quadrilateral bisect each other, the opposite sides are equal.

Ex. **37**. The radius of a sphere is 9 in. Find the volume of a spherical wedge whose angle is equal to 60°.

Ex. **38**. Find the radius of a sphere equivalent to a cone of revolution, the radius of whose base is equal to r and whose altitude is equal to h.

Ex. **39**. The area of a zone is equal to A; its altitude is equal to h. Find the radius of the sphere.

Ex. **40**. The volume of a sphere is numerically equal to one half its surface. Find the radius.

Ex. **41**. The volume of a cylinder of revolution is equal to one half the product of its lateral surface by the radius of its base.

Ex. **42**. What fraction of the surface of a sphere is visible to an observer whose distance from the center equals three times the radius?

Ex. **43**. How many square miles of the surface of the earth can be seen from a point 1000 mi. above the surface, if the earth is supposed to be a perfect sphere whose radius is equal to 4000 mi.?

Ex. **44**. How far from the center of a sphere whose radius is 6 ft. is an observer to whom $\frac{5}{12}$ of the entire surface is visible?

Ex. **45**. If from a point without a sphere a tangent and a secant be drawn, the tangent is the mean proportional between the secant and its external segment.

Ex. **46**. Through a sphere whose diameter is 10 in. a cylindrical hole of 6 in. diameter is bored. Find the volume of the solid if the axis of the cylinder passes through the center of the sphere.

Ex. **47**. The radius of a sphere is r, the area of a small circle a. Find its distance from the center.

Ex. **48**. The volume of a sphere is V. Find the surface of an equilateral spherical triangle whose angle is equal to 100°.

Ex. **49**. Find the area of the torrid zone if its altitude is $\frac{4}{5}$ of the radius of the earth.

Ex. **50**. How many miles from the surface of the earth can $\frac{1}{20}$ of the surface be seen?

Ex. **51**. The average pressure of the atmosphere upon each square inch of the surface is 15 lb. Find the total weight of the atmosphere.

Ex. **52**. How many bullets $\frac{1}{4}$ of an inch in diameter can be made from a cubic foot of lead?

Ex. **53**. A coffee pot is 8 in. high, 4 in. in diameter at the top, and 5 in. in diameter at the bottom. How many cups of coffee, each having a capacity of 10 cu. in., will the pot hold?

Ex. **54**. The places, A, B, and C, on the surface of the earth determine a spherical triangle ABC, whose angles are: $A = 50°$, $B = 61°$, $C = 71°$. Find the area of triangle ABC.

REVIEW EXERCISES

Ex. **1**. The sum of the exterior angles of a convex spherical polygon is less than four right angles.

Ex. **2**. The sum of the squares on three concurrent chords of a sphere at right angles to one another is constant for a fixed point.

Ex. **3**. What are the conditions in order that a spherical triangle may be self-polar?

Ex. **4**. A sphere can be inscribed in a right circular cylinder, and will touch it along a circle.

Ex. **5**. What is the locus a point such that the sum of the squares of its distances from two fixed points is constant?

Ex. **6**. The radius of a sphere is 12 inches. Find the circumference of the small circle made by the plane that bisects a radius at right angles.

Ex. 7. A cylindrical hole is bored through the center of a sphere. The remaining volume equals that of a sphere having its diameter equal to the length of the hole.

Ex. 8. What relation exists between the area of a spherical triangle and the perimeter of its polar triangle?

Ex. 9. The base of a right circular cone is a circle of a sphere. Prove the cone intersects the sphere in another circle.

Ex. 10. Find the volume generated by revolving an equilateral triangle about one of its sides as an axis.

Ex. 11. Two intersecting planes tangent to a cylinder intersect in a line parallel to the elements.

Ex. 12. The altitude of a regular tetrahedron is equal to the sum of the four perpendiculars drawn from any point within it upon its four faces.

Ex. 13. Prove: The mid-points of the edges of a regular tetrahedron are the vertices of a regular octahedron.

Ex. 14. Prove: The opposite trihedral angles of a parallelopiped are congruent.

Ex. 15. Prove: If the four diagonals of a parallelopiped are equal, the figure is a rectangular parallelopiped.

Ex. 16. The shortest distances between two parallel planes is the perpendicular to these planes.

Ex. 17. If the projections of a line on each of two intersecting planes be straight, the line itself must be straight with one exception. State the exception.

Ex. 18. Prove: The projection of a circle will be a circle only in case the circle is parallel to the plane upon which it is projected.

Ex. 19. A regular octahedron is cut by a plane parallel to one of its faces. Show that the perimeter of the section is constant.

Ex. 20. A sphere is inscribed in a right circular cylinder. Prove that the ratio of their total areas is equal to the ratio of their volumes.

Ex. 21. The volumes of two polyhedrons circumscribed about two equal spheres are proportional to their areas.

Ex. 22. Parallel planes cut a sphere in circles that have the same poles.

Ex. **23**. Two planes perpendicular respectively to two nonparallel lines must intersect. What is true if the lines are parallel?

Ex. **24**. Planes perpendicular respectively to the faces of a dihedral angle must intersect unless they are both $\perp$ to the edge of the $\angle$.

Ex. **25**. Prove that the locus of the feet of the perpendiculars drawn from a fixed point on planes passing through a fixed point is a sphere.

Ex. **26**. Prove that the locus of the feet of the perpendiculars from a fixed point to planes passing through a fixed line is a circle.

Ex. **27**. The projections of a line upon two parallel lines are equal.

Applying the principles developed on pages 301–304, solve the following:

Ex. **28**. How many degrees are there in the angle between a diagonal of a cube and its projection upon a face of the cube.

Ex. **29**. How many degrees are there in the dihedral angle formed by two faces of a regular tetrahedron?

Ex. **30**. Find the volume of a cone of revolution if the hypotenuse of the revolving right triangle forms an angle of 40° with the axis, and the hypotenuse is 12 in.

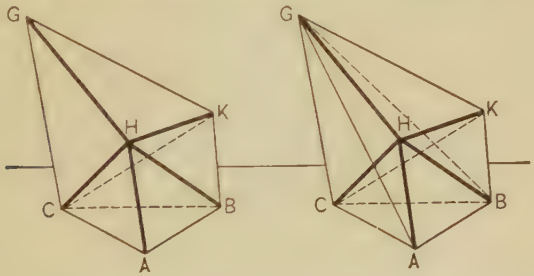
Ex. **31**. Find the volume of a regular quadrangular pyramid, given each side of the base is 6 feet and the dihedral angle made by each face with the base is 36°.

Ex. **32**. Find the volume of a circular cylinder having the radius of the base 4 feet and each element (10 feet long) forming an angle of 25° with the base.

APPENDIX TO SOLID GEOMETRY

Proposition I. Theorem

808. *A truncated triangular prism is equivalent to the sum of three pyramids whose common base is the base of the prism and whose vertices are the three vertices of the inclined section.*



Given ABC–HKG, a truncated triangular prism of which ABC is the base.

To prove ABC–$HGK = H$–$ABC + K$–$ABC + G$–ABC.

Proof. Pass planes through H, B, C, and through H, K, C, forming the pyramids H–ABC, H–BCK, and H–CKG.

H–ABC is evidently one of the required pyramids.

H–BCK may be read C–HBK.

$$C\text{–}HBK = C\text{–}AKB. \tag{637}$$

C–AKB may be read K–ABC.

$\therefore$ H–BCK is equivalent to the second of the required pyramids.

$$H\text{–}CKG = H\text{–}GBC. \tag{637}$$

H–GBC or B–HGC = B–AGC or G–ABC.

Hence H–CKG = the third of the required pyramids.

$\therefore$ ABC–HKG = H–ABC + K–ABC + G–ABC.

809. Cor. 1. *The volume of a truncated right triangular prism is equal to the product of its base by one third the sum of its lateral edges.*

810. Cor. 2. *The volume of any truncated triangular prism is equal to the product of its right section by one third the sum of its lateral edges.*

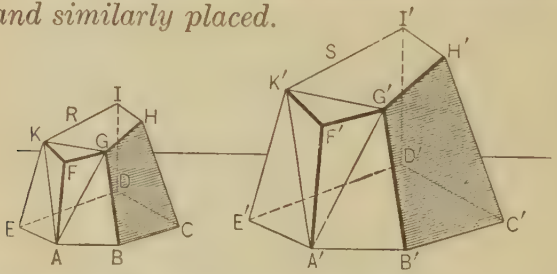
Hint. The right section divides the truncated prism so that two truncated right prisms are formed.

Ex. 1. Find the volume of a truncated right triangular pyramid, if the lateral edges are 4, 5, and 6 and the edges of the base are 6, 8, and 10.

Ex. 2. The sides of the base of a truncated triangular prism are 13, 14, and 15, and the lateral edges are 6, 8, and 10. Find the volume if the inclination of the edges to the base is 30°.

Proposition II. Theorem

811. *Two similar polyhedrons can be decomposed into the same number of tetrahedrons similar each to each and similarly placed.*



Given R and S, two similar polyhedrons; A and A', corresponding vertices.

To prove R and S can be decomposed into the same number of tetrahedrons, similar each to each, and similarly placed.

Proof. From A and A' draw all diagonals of the two solids.

If we consider every face of R and S, except those that pass through A or A', as the base of a pyramid whose vertex is A or A', each solid is decomposed into as many pyramids as we have assumed bases.

By drawing in the corresponding bases, the corresponding diagonals, and passing planes through these diagonals and A or A' respectively, we decompose corresponding pyramids into the same number of triangular pyramids.

Hence, R and S can be decomposed into the same number of tetrahedrons.

In tetrahedrons A–KFG and A'–$K'F'G'$,

$$\left.\begin{array}{l} \triangle AKF \sim \triangle A'K'F' \\ \triangle AFG \sim \triangle A'F'G' \\ \triangle KFG \sim \triangle K'F'G' \end{array}\right\} \quad (315)$$

$$\frac{AK}{A'K'} = \frac{KF}{K'F'} = \frac{KG}{K'G'} = \frac{FG}{F'G'} = \frac{AG}{G'A'}. \quad \text{(Why?)}$$

$$\therefore \triangle AKG \sim \triangle A'K'G'. \quad \text{(Why?)}$$

The tetrahedrons A–KFG and A'–$K'F'G'$ have all corresponding trihedral angles equal. (559)

$\therefore$ tetrahedron A–KFG is similar to tetrahedron A'–$K'F'G'$. (641)

When A–KFG and A'–$K'F'G'$ are removed, the remaining polyhedrons are similar, for their faces remain similar, and the polyhedral angles are equal.

In like manner any other two corresponding tetrahedrons can be proved similar.

Hence R and S can be decomposed into the same number of tetrahedrons similar each to each and similarly placed.

812. Cor. 1. *The corresponding edges of similar polyhedrons are proportional.*

813. Cor. 2. *Any two corresponding lines in two similar polyhedrons have the same ratio as any two corresponding edges.*

814. Cor. 3. *Two corresponding faces of similar polyhedrons are to each other as the squares of any two corresponding edges.*

815. Cor. 4. *The total areas of two similar polyhedrons are to each other as the squares of any two corresponding edges.*

816. Cor. 5. *The volumes of two similar polyhedrons are to each other as the cubes of any two corresponding edges.*

Hint. If an edge of the first polyhedron is to the corresponding edge of the second polyhedron as $1 : n$, then any edge of the second polyhedron is n times the corresponding edge of the first polyhedron. Hence any tetrahedron which forms a part of the second polyhedron is n^3 times the corresponding tetrahedron of the polyhedron. Hence by addition we obtain the conclusion.

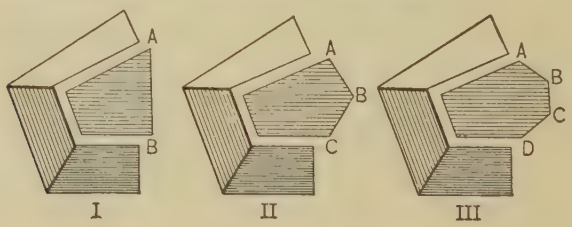
Ex. 1. Prove that all regular dodecahedrons are similar.

Ex. 2. If R and S are the volumes of two similar polyhedrons, and r and s two corresponding edges, find S if $R = 100$ cu. in., $r = 5$, and $s = 4$.

Ex. 3. The volumes of two similar polyhedrons are 216 cu. in. and 343 cu. in. If an edge of the first figure equals 12 in., find the corresponding edge of the second figure.

Proposition III. Theorem

817. *In any convex polyhedron, the number of edges increased by two, equals the sum of the number of vertices plus the number of faces.* (Euler's Theorem.)



Given E, the number of edges, V, the number of vertices, and F, the number of faces of a convex polyhedron.

To prove $V + F = E + 2.$

Proof. Imagine the solid formed by laying down one face, and then adding successively all the other faces, and denote the *increase* of the number of vertices, faces, and edges produced by the addition of the mth face by v_m, f_m, e_m.

Obviously $f_m = 1$, no matter what the value of m.

If $e_m = 1$, $v_m = 0$. (Fig. I)

If $e_m = 2$, $v_m = 1$. (Fig. II)

If $e_m = 3$, $v_m = 2$. (Fig. III), etc.

It is not difficult to see that in general $e_m - v_m = 1 = f_m$.

Or $$v_m + f_m - e_m = 0.$$

Hence $V + F - E$ is in general not changed by the addition of any face. There are, however, two exceptions, viz. the first face which is laid down, and the last one.

Obviously $$v_1 = e_1.$$

Therefore $$v_1 + f_1 - e_1 = 1.$$

And since for the last (or nth) face, $v_n = 0$, $e_n = 0$, we have $$v_n + f_n - e_n = 1.$$

Hence we have

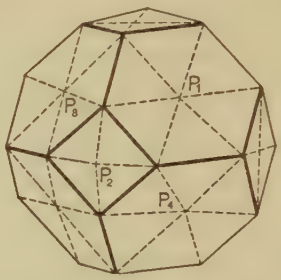
$$v_1 + f_1 - e_1 = 1.$$
$$v_2 + f_2 - e_2 = 0.$$
$$v_3 + f_3 - e_3 = 0.$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$v_n + f_n - e_n = 1.$$

By adding we obtain

$$V + F - E = 2.$$
$$\therefore V + F = E + 2.$$

Proposition IV. Theorem

818. *The sum of the face angles of a convex polyhedron of n vertices is equal to $2(n-2)$ straight angles.*



Given n, the number of vertices, and s, the sum of the face angles of a convex polyhedron.

To prove $s = 2(n-2)$ st. $\measuredangle$.

Proof. In every face join a point (as P_1, P_2, P_3, etc.) to all vertices of the face.

Let F = the number of faces of the polyhedron.

E = the number of edges of the polyhedron.

P = the sum of all angles formed at P_1, P_2, P_3, etc.

T = the sum of all angles of the triangles formed in the faces.

Obviously $s = T - P$.

Since every edge is the base of two triangles, we have $2E$ triangles, and

$$T = 2E \text{ st. } \measuredangle.$$

Since the number of points, P_1, P_2, P_3, etc. is F,

$$P = F \times 2 \text{ st. } \measuredangle = 2F \text{ st. } \angle.$$

$$\therefore s = (2E - 2F) \text{ st. } \measuredangle$$

$$= 2(E - F) \text{ st. } \measuredangle.$$

But it follows easily from the preceding proposition that

$$E - F = n - 2 \text{ (for } n = V).$$

$$\therefore s = 2(n - 2) \text{ st. } \measuredangle.$$

Ex. 1. Find the sum of the face angles of a pyramid of ten lateral faces.

Ex. 2. Find the sum of the face angles of a polyhedron with six vertices; with 10 vertices.

Ex. 3. Find the sum of the face angles of a polyhedron, if the number of its edges is 30 and the number of its faces is 12.

Ex. 4. Show that the number of faces, vertices, and edges of the regular solids conform with Euler's theorem.

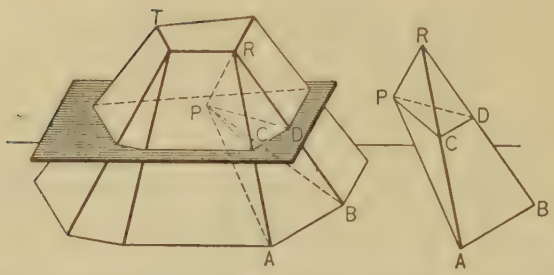
Ex. 5. Prove that there can be no polyhedron with 7 edges.

819. DEF. A **prismatoid** is a polyhedron bounded by two polygons (called *bases*) in parallel planes and by a number of triangles and trapezoids (called *lateral faces*) with one side lying in one base and the opposite vertex or side lying in the other base.

The *altitude* is the distance between the planes of the bases. The *mid-section* is the section made by a plane parallel to the bases and bisecting the altitude.

PROPOSITION V. THEOREM

820. *The volume of a prismatoid is equal to the sum of its bases and four times its mid-section multiplied by one sixth of its altitude.*



Given V, the volume; H, the altitude; B and b, the respective areas of the bases; and M, the area of the mid-section of prismatoid ABT.

To prove $$V = \frac{H}{6}(B + b + 4M).$$

Proof. If a lateral face is a trapezoid, divide it into two triangles by drawing the diagonal.

Let P be any point in M, and join P to all vertices. Then the solid is divided into a number of pyramids, all having P as their vertex, and for their bases B, b, and the triangles formed by the lateral faces.

$$\text{The volume of } P\text{–}B = \frac{1}{3} \cdot \frac{H}{2} \cdot B = \frac{H}{6} \cdot B.$$

$$\text{The volume of } P\text{–}b = \frac{1}{3} \cdot \frac{H}{2} \cdot b = \frac{H}{6} \cdot b.$$

To find the volume of the remaining pyramids let us consider the pyramid P–ABR.

$$RC = CA,\ RD = DB. \tag{496}$$

$$\therefore \triangle RAB = 4(\triangle RCD).$$

$$\therefore P\text{–}RAB = 4(P\text{–}RCD). \tag{636}$$

But if we consider R as vertex of pyramid P–RCD, its volume is obviously $\frac{H}{6}(\triangle PCD)$.

$$\therefore P\text{–}RAB = 4 \cdot \frac{H}{6}(\triangle PCD).$$

In like manner the volume of each lateral pyramid is equal to $4 \cdot \frac{H}{6} \times$ the area of that portion of M which is included by the pyramid.

Hence the sum of all lateral pyramids equals $4 \cdot \frac{H}{6} \cdot M$.

$$\therefore V = \frac{H}{6}B + \frac{H}{6}b + 4\frac{H}{6}M,$$

or

$$V = \frac{H}{6}(B + b + 4M).$$

NOTE. By means of the calculus it can be proved that the volumes of the solids in elementary geometry can all be obtained by using the prismatoid formula.

Ex. Find the volume of a sphere having its radius equal to R.

Sol. By placing the sphere between two parallel tangent planes and considering the two bases as circles of radii zero, we obtain:

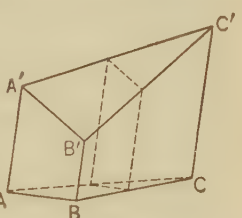
$$V(\text{sphere}) = \tfrac{1}{6} \times 2R \times (0 + 0 + 4 \times \pi R^2) = \tfrac{4}{3}\pi R^3.$$

Ex. **1**. Find the volume of (*a*) a circular cylinder, (*b*) a circular cone, (*c*) the frustum of a circular cone.

Ex. **2**. Derive from the prismatoid formula, the formula for the volume of (*a*) a prism, (*b*) a pyramid, (*c*) a frustum of a pyramid.

Ex. **3**. Find the volume of a prismatoid if $B = 20$, $b = 6$, $M = 12$, and $H = 15$.

Ex. **4**. In the truncated prism $ABC\text{–}A'B'C'$, $AA' = 6$, $BB' = 2$, $CC' = 8$, the altitude of trapezoid $A'B = 4$, and the perpendicular from C upon $ABB'A' = 12$. Find the volume of the solid.



Ex. 4

LOGARITHM TABLE

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
N	0	1	2	3	4	5	6	7	8	9

LOGARITHM TABLE

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996
N	0	1	2	3	4	5	6	7	8	9

TABLES OF SQUARES, CUBES, AND SQUARE AND CUBE ROOTS

x	x^2	x^3	$\sqrt{x}$	$\sqrt[3]{x}$	x	x^2	x^3	$\sqrt{x}$	$\sqrt[3]{x}$
1	1	1	1.000	1.000	2 6	6 76	17 576	5.099	2.962
2	4	8	1.414	1.260	2 7	7 29	19 683	5.196	3.000
3	9	27	1.732	1.442	2 8	7 84	21 952	5.292	3.037
4	16	64	2.000	1.587	2 9	8 41	24 389	5.385	3.072
5	25	125	2.236	1.710	3 0	9 00	27 000	5.477	3.107
6	36	216	2.449	1.817	3 1	9 61	29 791	5.568	3.141
7	49	343	2.646	1.913	3 2	10 24	32 768	5.657	3.175
8	64	512	2.828	2.000	3 3	10 89	35 937	5.745	3.208
9	81	729	3.000	2.080	3 4	11 56	39 304	5.831	3.240
10	1 00	1 000	3.162	2.154	3 5	12 25	42 875	5.916	3.271
11	1 21	1 331	3.317	2.224	3 6	12 96	46 656	6.000	3.302
12	1 44	1 728	3.464	2.289	3 7	13 69	50 653	6.083	3.332
13	1 69	2 197	3.606	2.351	3 8	14 44	54 872	6.164	3.362
14	1 96	2 744	3.742	2.410	3 9	15 21	59 319	6.245	3.391
15	2 25	3 375	3.873	2.466	4 0	16 00	64 000	6.325	3.420
16	2 56	4 096	4.000	2.520	4 1	16 81	68 921	6.403	3.448
17	2 89	4 913	4.123	2.571	4 2	17 64	74 088	6.481	3.476
18	3 24	5 832	4.243	2.620	4 3	18 49	79 507	6.557	3.503
19	3 61	6 859	4.359	2.668	4 4	19 36	85 184	6.633	3.530
20	4 00	8 000	4.472	2.714	4 5	20 25	91 125	6.708	3.557
21	4 41	9 261	4.583	2.759	4 6	21 16	97 336	6.782	3.583
22	4 84	10 648	4.690	2.802	4 7	22 09	103 823	6.856	3.609
23	5 29	12 167	4.796	2.844	4 8	23 04	110 592	6.928	3.634
24	5 76	13 824	4.899	2.884	4 9	24 01	117 649	7.000	3.659
25	6 25	15 625	5.000	2.924	5 0	25 00	125 000	7.071	3.684

USEFUL VALUES

1 cu. ft. = 1728 cu. in.

1 gal. = 231 cu. in.

1 bu. = 2150.4 cu. in.

$\pi = 3.1416$

$2\pi = 6.2832$

$\frac{\pi}{4} = 0.7854$

$\frac{\pi}{6} = 0.5236$

$\pi^2 = 9.8696$

$\sqrt{\pi} = 1.7725$

$\sqrt[3]{\pi} = 1.4646$

$\frac{1}{\pi} = 0.3183$

$\frac{1}{\sqrt{\pi}} = 0.5642$

$\frac{\pi}{180} = 0.0175$

FORMULAS IN SOLID GEOMETRY

L = lateral area; S = slant height; V = volume; E = lateral edge (also spherical excess); B and b = bases; T = total area.

Areas of Solid Figures:

Right prism. $L = E \times$ perimeter of base.

Any prism. $L = E \times$ perimeter of a right section.

Regular pyramid, $L = \frac{1}{2} S \times$ perimeter of base.

Frustum of regular pyramid, $L = \frac{1}{2} S \times$ sum of perimeters of the two bases.

Cylinder of revolution, $L = 2\pi RH$, $T = 2\pi R(H + R)$.

Cone of revolution, $L = \pi RS$, $T = \pi R(S + R)$.

Frustum of cone of revolution, $L = \pi(R + R')S$;

$T = L + \pi(R^2 + R'^2)$.

Sphere, $T = 4\pi R^2$.

Zone, $T = 2\pi RH$.

Lune, $T = \dfrac{\angle A^\circ}{90^\circ} \times \pi R^2$.

Spherical $\triangle$, $T = \dfrac{E^\circ}{180^\circ} \times \pi R^2$.

Excess spherical $\triangle$, $E = (\angle A^\circ + \angle B^\circ + \angle C^\circ) - \angle 180^\circ$.

Volumes:

Rectangular parallelopiped, V = product of the 3 dimensions.

Parallelopiped, or prism, or cylinder, $V = B \times H.$

Pyramid, or cone, $V = \frac{1}{3} B \times H.$

Frustum of pyr. or cone, $V = \frac{1}{3} H(B + b + \sqrt{B \times b}).$

Cyl. of revolution, $V = \pi R^2 H.$

Cone of revolution, $V = \frac{1}{3} \pi R^2 H.$

Frustum of c. of revolution, $V = \frac{1}{3} \pi H(R^2 + R'^2 + RR').$

Prismatoid, $V = \frac{H}{6}(B + b + 4\,M).$

Sphere, $V = \frac{4}{3} \pi R^3 = \frac{\pi}{6} \times (\text{Diam.})^3.$

Spherical pyramid, $V = \frac{1}{3} B \times R.$

Spherical sector, $V = \frac{1}{3} R \times$ zone.

Spherical segment, $V = \frac{\pi H}{2}(R^2 + R'^2) + \frac{\pi H^3}{6}.$

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